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Stability of bifurcating periodic orbits: an application to laser equations

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Abstract

Based on the Poincarè-Lindstedt perturbation method, we propose a general analytical procedure to determine the stability of periodic solutions arising from a Hopf bifurcation in dynamical systems. As an application of our method to a physical system, we analyze the stability of bifurcating periodic orbits in a single mode laser. An analytic expression for the associated stability coefficient is obtained and the stability regions are characterized in the space of parameters of this system.

Key words: Hopf bifurcation; limit cycles; nonlinear dynamical systems; single mode laser.

Estabilidad de órbitas periódicas bifurcantes: una aplicación a ecuaciones de láser

Resumen

Con base en el método perturbativo de Poincarè-Lindstedt, se propone un procedimiento analítico general para determinar la estabilidad de soluciones periódicas originadas a partir de una bifurcación de Hopf en sistemas dinámicos. Como una aplicación de nuestro método a un sistema físico, analizamos la estabilidad de órbitas periódicas bifurcantes en las ecuaciones del láser de modo simple. Se obtiene una expresión analítica para el coeficiente de estabilidad, y las regiones de estabilidad se caracterizan en el espacio de parámetros de este sistema.

Palabras clave: Bifurcación de Hopf, ciclos límites, láser de modo simple; sistemas dinámicos no lineales.

1. Introduction

The emergence of oscillations as a parameter is varied is a phenomenon that occurs widely in non-equilibrium systems in nature. This phenomenon can very often be described by a Hopf bifurcation (1). Biological, chemical, physical, social and economic systems possess this property and, as a consequence, they may exhibit self-sustained

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oscillations in the absence of external periodic forces. Some relevant examples are: the beating of a heart, the periodic firing of neurons, daily rhythms in the human body temperature and hormone secretion, oscillating chemical reactions, economic cycles, and dangerous vibrations in bridges and airplane wings (2).

In this context, the study of the stability of the emerging oscillations is currently an important problem which carries a long tradition. Poincarè (3) was the first to study the stability of periodic orbits or limit cycles. Hopf (4) investigated the asymptotic stability of the bifurcating orbits by using the Poincarè perturbation method for constructing periodic solutions and Floquet theory of equations (5) with periodic coefficients. La Salle (6) used the Poincarè-Bendixon theorem to prove the existence of a stable limit cycle containing the origin for the general van der Pol's equations. Subsequent approaches include outgrowths of earlier works, topological methods such as the Center Manifold Theorem (7, 8) and frequency methods (9). There are also available algorithms to compute numerically the stability of the periodic orbits. The analytical calculations are usually very involving so that in most applications the work has been carried out numerically.

The purpose of this paper is to present a simple analytical method to determine the stability of periodic solutions arising from a Hopf bifurcation. Our procedure is based on the Poincarè-Lindstedt perturbation method, which has found specific applications to several physical systems (10-12). We consider that this approach is more appealing and intuitive to physicists. It is presented in Section 2.

As an application of our procedure to a physical system, in Section 3 we analyze the stability of bifurcating time-periodic orbits in a single mode laser. We have obtained an analytic expression for the associated stability coefficient in this case.

In the Conclusions we discuss the proposed method and compare the results of the single mode laser with the studied facts of the Lorenz system. Some further applications and extensions of the method are suggested.

2. Stability of bifurcating time-periodic orbits

Many phenomena can be modeled in terms of systems of autonomous differential equations:

$$\frac{dx}{dt} = f(x,\lambda), \quad x \in \Re^n, \qquad [2.1]$$

where λ is a parameter (or set of parameters) on a real interval. Let $x_s(\lambda)$ be a stationary or fixed point of the system [2.1], i.e.,

$$f\big(x_s(\lambda),\lambda) = 0\big). \qquad [2.2]$$

We wish to delineate the circumstances under which the steady solution $x_s(\lambda)$ of the system [2.1] loses its stability to a time-periodic motion

$$x(t) = x_s(\lambda) + z(\lambda,t), \qquad [2.3]$$

where

$$z(\lambda,t) = z\left(\lambda, t + \frac{2\pi}{\omega(\lambda)}\right). \qquad [2.4]$$

and we want to determine the stability conditions for the resulting periodic orbit $z(\lambda,t)$. Assuming that the deviations $z(t)$ are small, we may write

$$\frac{dz(\lambda,t)}{dt} = f(x_s + z,\lambda)$$

$$= L(\lambda)z(\lambda,t) + N(\lambda,t), \qquad [2.5]$$

where the Jacobian matrix $L(\lambda)$ has components

$$L_{ij}(\lambda) = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_s(\lambda)}, \quad (i, j = 1,\ldots,n);$$ [2.6]

and $N(\lambda,t)$ represents the non-vanishing non-linear terms. To second order,

$$N_i(\lambda,t) = \frac{1}{2}\left.\frac{\partial f_1}{\partial x_j \partial x_k}\right|_{x_s(\lambda)} z_j z_k .$$ [2.7]

We denote by $\alpha_1(\lambda)$ the eigenvalue corresponding to the eigenvector u_i of $L(\lambda)$:

$$Lu_i = \alpha_i(\lambda)u_i \quad , \quad (i = 1,\ldots,n)$$ [2.8]

Suppose that a Hopf bifurcation occurs at $\lambda = \lambda_0$. The Hopf bifurcation is characterized by a change of stability of the stationary point $x_s(\lambda)$ accompanied by the creation of a closed trajectory or limit cycle (i.e. periodic solutions) in the phase space $x(\lambda)$. There are several versions of the theorem on the explicit conditions for such bifurcation to occur. Here we shall use it in the following form (1):

a. Suppose that two eigenvalues $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ of L are a complex conjugate pair $\left(\alpha_1(\lambda) = \alpha^*_{21}(\lambda)\right)$.

b. Assume that for $\lambda < \lambda_0$, Re $\alpha_1(\lambda) < 0$; for $\lambda > \lambda_0$, Re $\alpha_1(\lambda) > 0$, for $\lambda = \lambda_0$, Re $\alpha_1(\lambda_0) = 0$. Im $\alpha_1(\lambda_0) \equiv \Omega \neq 0$.

c. Assume that $\left.\frac{d\,\mathrm{Re}\,\alpha_1(\lambda)}{d\lambda}\right|_{\lambda_0} > 0$; and that Re $\alpha_1(\lambda_0) = 0$ $(j = 3,\ldots,n)$.

d. The there is a closed orbit in a neighborhood of $x_s(\lambda)$ with approximate period $\frac{2\pi}{\Omega}$ and growing as $|\lambda|^{1/2}$.

Periodic solutions which exist for $\lambda > \lambda_0$ are named **supercritical** and those occurring for $\lambda < \lambda_0$ are called **subcritical**. After a Hopf bifurcation has taken place one may ask about the stability of the resulting periodic orbit. In this Section we propose an analytical method to show that if the Hopf bifurcation is supercritical, the closed orbits are stable under small perturbations and, if the bifurcations if subcritical, the orbits are unstable.

Let the two purely imaginary eigenvalues of $L(\lambda)$ at the value of the parameter $\lambda = \lambda_0$ be

$$\alpha_1(\lambda_0) = \alpha^*_{21}(\lambda_0) = i\Omega .$$ [2.9]

We seek periodic solutions of [2.5] by introducing a scaled time τ and a 2π-periodic function y through

$$\tau = \omega(\varepsilon)t ,$$

$$z(t) = \varepsilon y(\tau, \varepsilon) ,$$

$$y(\tau, \varepsilon) = y(\tau + 2\pi, \varepsilon) ,$$

$$\lambda = \lambda(\varepsilon) ,$$ [2.10]

such that as $\varepsilon \to 0$, $\omega(\varepsilon) \to \Omega$, $\lambda \to \lambda_0$, $z(t) \to 0$.

Then we can write equation [2.5] as:

$$\omega(\varepsilon)\frac{\partial y(\varepsilon,t)}{\partial \tau} = L\left(\lambda(\varepsilon)\right)y(\tau,\varepsilon) + \varepsilon N\left(\lambda(\varepsilon),\tau\right)$$ [2,11]

assuming N of second order.

2.1 Adjoint problem for the homogeneous equation

Letting $\varepsilon \to 0$ we see that $y(\tau,0)$ must satisfy

$$Jy(\tau,0) = 0 ,$$ [2.12]

with

$$J \equiv \Omega\frac{\partial}{\partial \tau} - L_0 ,$$ [2.13]

where

$$L_0 \equiv L(\lambda_0) .$$ [2.14]

The solution of [2.12] can be expressed as

$$y(\tau,0)=\exp\left(\frac{L_0}{\Omega}\tau\right)u\,, \qquad [2.15]$$

where u is an eigenvector of L_0 with eigenvalue $i\Omega$,

$$L_0 u = i\Omega u\,. \qquad [2.16]$$

Then the solution of [2.12] may be written in the form

$$y(\tau,0)=e^{i\tau}u+e^{-i\tau}u^*\,. \qquad [2.17]$$

Define the scalar product in $\Re^n$ as:

$$\langle a|b\rangle \equiv \sum_{i=0}^{n} a_i b_i^*\,, \qquad [2.18]$$

and the adjoint matrix of L_0 by

$$\langle aL_0|b\rangle = \langle M_0 a|b\rangle\,, \qquad [2.19]$$

then

$$M_{0ij} = L_{0ij}^*\,. \qquad [2.20]$$

Consider the eigenvector ν of M_0 with eigenvalue $i\Omega$:

$$M_0\nu = i\Omega\nu\,, \qquad [2.21]$$

then the adjoint homogeneous equation to [2.12] is

$$Iw = 0\,, \qquad [2.22]$$

where

$$I \equiv \Omega\frac{\partial}{\partial\tau}+L_0\,, \qquad [2.23]$$

whose solution can be expressed as

$$w = e^{-i\tau}\nu + e^{i\tau}\nu^*\,. \qquad [2.24]$$

For future use, we define the vectors

$$\begin{array}{ll} \eta_1 = e^{i\tau}u & \tilde{\eta}_1 = e^{i\tau}\nu^* \\ \eta_2 = e^{-i\tau}u^* & \tilde{\eta}_2 = e^{-i\tau}\nu \end{array}, \qquad [2.25]$$

which satisfy

$$J\eta_i = 0, \quad I\tilde{\eta}_i = 0; \quad i = 1,2\,. \qquad [2.26]$$

We choose the normalization

$$\langle u|\nu^*\rangle = 1\,. \qquad [2.27]$$

Finally, let us introduce a scalar product for time-dependent 2π-periodic vectors $a(\tau)$, $b(\tau)$:

$$(a|b) \equiv \frac{1}{2\pi}\int_0^{2\pi}\langle a(\tau)|b(\tau)\rangle d\tau\,. \qquad [2.28]$$

The parameter ε will be defined by the implicit equation

$$\varepsilon = \left(y(\tau,\varepsilon)|\nu^*\right), \qquad [2.29]$$

and it is related with the amplitude of the motion.

2.2. Construction of the time-periodic solution

We look for a solution of [2.11] as a Taylor expansion in ε:

$$y(\tau,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n y(\tau,\varepsilon)\,,$$

$$\omega(\varepsilon) = \Omega + \sum_{n=1}^{\infty}\varepsilon^n\omega_n\,,$$

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty}\varepsilon^n\lambda_n\,,$$

$$L(\lambda(\varepsilon)) = L_0 + \sum_{n=1}^{\infty}\varepsilon^n L_n(\lambda_n)\,. \qquad [2.30]$$

Substitution in [2.11] yields ($n = 0$)

$$\left(\Omega \frac{\partial}{\partial\tau} - L_0\right) y(\tau,0) = Jy(\tau,0) = 0\,, \quad [2.31]$$

and $(n > 0)$

$$Jy(\tau,n) = \varphi_n(\tau)\,, \quad [2.32]$$

where

$$\varphi_n(\tau) \equiv \sum_{m=0}^{\infty} L_{n-m}(\lambda_n) y(\tau,m) - \omega_n \frac{\partial y(\tau,0)}{\partial\tau} + eN(\tau,\varepsilon^{n-1})\,. \quad [2.33]$$

The function $\varphi_n(\tau)$ depends on τ via the $y(\tau,n')$, with $n' < n$.

Equation [2.32] is a first-order inhomogeneous linear differential equation. Its general solution is the sum of the solution of homogeneous equation $JY = 0$ and of a particular solution of the inhomogeneous equation. The adjoint of the homogeneous equation is:

$$I\omega(\tau) = 0\,, \quad [2.34]$$

whose 2π-periodic solutions are $\tilde{\eta}_1$ and $\tilde{\eta}_2$, since $I\tilde{\eta}_1 = 0$. Forming the scalar product of $I\tilde{\eta}_1$ with $y(\tau,n)$ leads to:

$$\begin{aligned} 0 &= \left(y(\tau,n) | I\tilde{\eta}_i\right), \\ &= \left(Jy(\tau,n) | \tilde{\eta}_i\right), \\ &= \left(\varphi_n(\tau) | \tilde{\eta}_i\right). \end{aligned} \quad [2.35]$$

To adhere to the standard notation, we write

$$[\varphi_n]_j \equiv \left(\varphi_n(\tau) | \tilde{\eta}_j\right) = 0\,. \quad [2.36]$$

Hence to solve equation [2.32], the inhomogeneous term must verify the solvability condition [2.36]. The structure of [2.36] indicates that only terms of $\varphi_n(\tau)$ proportional to $\exp(\pm i\,\tau)$ will contribute to the solvability condition. These secular terms lead to an aperiodic contribution to $y(\tau,n)$ proportional to $\tau^2 \exp(\pm i\,\tau)$, $(s > 0)$. The requirement that such contributions identically vanish is equivalent to the solvability condition [2.36].

Equations [2.32] can be solved sequentially, providing at each stage with equations for λ_n and ω_n:

$$\left[\varphi_n(\tau)\right]_1 = -i\omega_n + \xi_n \lambda_n + \left[N_{n-1}(\tau)\right]_1 = 0\,, \quad [2.37]$$

where

$$\xi_n \equiv \left[\sum_{m=0}^{\infty} L_{n-m} y(m,\tau)\right]_1, \quad [2.38]$$

and $N_{n-1}(\tau)$ denotes nonlinear terms of order $n-1$. Since we want ω_n, λ_n to be real:

$$\begin{aligned} \omega_n &= -\mathrm{Im}\left\{\xi_n \lambda_n + [N_{n-1}]_1\right\}, \\ \lambda_n \,\mathrm{Re}\,\xi_n &= -\mathrm{Re}[N_{n-1}]_1\,. \end{aligned} \quad [2.39]$$

With these values of ω_n and λ_n, equation [2.32] may be solved and the solution is unique up to the solution of the homogeneous problem, which we can take always to be zero. This gives an additional constraint under which equation [2.32] has a unique solution.

We note that for $n = 1$, $[N_0] = 0$, so that:

$$\lambda_1 = 0, \omega_1 = 0\,. \quad [2.40]$$

Thus the first effects of the nonlinearity on $\lambda(\varepsilon)$ and $\omega(\varepsilon)$ occur at second order. In fact, it can be proved by induction that

$$\lambda_{2l+1} = \omega_{2l+1} = 0\,. \quad [2.41]$$

It follows from this that $\lambda(\varepsilon)$ and $\omega(\varepsilon)$ are even functions and the bifurcation is definitely subcritical $(\lambda < \lambda_0)$ or supercritical $(\lambda > \lambda_0)$.

2.3. Stability of the periodic motion

To second order we get $\lambda = \lambda_0 + \varepsilon^2 \lambda_2$. Thus the sign of λ_2 determines the critical character of the bifurcation. In Appendix I, we show that the application of the Floquet theory of stability of equations with periodic terms yields the following result: subcritical periodic motions ($\lambda_2 < 0$) are unstable and supercritical periodic motions ($\lambda_2 > 0$) are stable. If $\lambda_2 = 0$, one may need to calculate λ_4.

3. An application: single mode laser

The single mode homogeneously broadened ring laser in the rotating wave approximation and the slowly varying amplitude approximation is described by the following equations (13):

$$\frac{dE}{dt} = kE - kP$$

$$\frac{dP}{dt} = \gamma ED - \gamma P$$

$$\frac{dD}{dt} = \gamma_{\parallel}(\lambda + 1) - \gamma_{\parallel} D - \gamma_{\parallel} \lambda EP, \qquad [3.1]$$

where E and P are the normalized electric field amplitude and normalized polarization, respectively. D is the threshold parameter measuring the population inversion compared to its critical value ($D_c = 1$). K is the decay rate of the electric field in the cavity (or cavity loss). The polarization decay rate γ is the reciprocal of the coherence time (or transverse relaxation time). Finally, $\lambda = (D_0 - D_c)D_c$ is the normalized pump parameter, where D_0, D_c are the unsaturated and critical inversion, respectively. γ can be experimentally varied and will be chosen as the bifurcation parameter. In the above equations it is assumed that the phase differences are kept equal to zero.

It was pointed out by Haken (14) that equations [3.1] are equivalent to the Lorenz equations of fluid dynamics. Therefore, as it is well known for the Lorenz system, a limit cycle solution, chaotic behavior and a strange attractor are also possible in the single mode laser.

We proceed to study a Hopf bifurcation for the laser system [3.1] and to determine analytically the stability of the resulting periodic solution according to the method presented in Section 2.

3.1. Steady states

The system [3.1] has the following stationary states

$$C_1: \quad E = 0, \quad P = 0, \quad D = \lambda + 1.$$

$$C_{2,3}: \quad E^2 = P^2 = 1, \quad D = 1. \qquad [3.2]$$

The state C_1 has the following linearized matrix:

$$\begin{pmatrix} -k & k & 0 \\ \gamma(\lambda+1) & -\gamma & 0 \\ 0 & 0 & -\gamma_{\parallel} \end{pmatrix}, \qquad [3.3]$$

with eigenvalues

$$\omega_{1,2} = \frac{-(\gamma + k) \pm \sqrt{(\gamma + k)^2 + 4k\lambda}}{2}, \qquad [3.4]$$

$$\omega_3 = -\gamma_{\parallel}. \qquad [3.5]$$

For $\lambda < 0$, $\omega_{1,2,3} < 0$ so that the state C_1 is asymptotically stable. For $\lambda > 0$, $\omega_{2,3} < 0$, $\omega_1 > 0$ and C_1 is unstable. We see that since $\gamma > 0$, $k > 0$ and $\gamma_{\parallel} > 0$, the real part of $\omega_{1,2,3}$ never vanishes and no Hopf bifurcation is possible from C_1. States $C_{2,3}$ are the interesting ones for our purposes. The linearized matrix at the point ($E = P = D = 1$) is:

$$L = \begin{pmatrix} -k & k & 0 \\ \gamma & -\gamma & \gamma \\ -\gamma_{\parallel}\lambda & -\gamma_{\parallel}\lambda & -\gamma_{\parallel} \end{pmatrix}, \qquad [3.6]$$

and has characteristic polynomial:

$$\omega^3 + (k+\gamma+\gamma_{\|})\omega^2 + \gamma_{\|}(k+\gamma+\gamma\lambda)\omega + 2k\gamma\gamma_{\|}\lambda = 0. \quad [3.7]$$

(the point ($E = P = -1$, $D = 1$) leads to the same equation).

3.2. Hopf bifurcation

To search for one real root and two pure imaginary roots let the characteristic polynomial [3.7] be written

$$(\omega-\beta)(\omega-\beta)(\omega-\alpha) = 0. \quad [3.8]$$

where

$$\beta = \beta_1 + i\beta_2, \quad [3.9]$$

thus,

$$\omega^3 - (2\beta_1+\alpha)\omega^2 + \left(|\beta|^2 + 2\beta_1\alpha\right)\omega - |\beta|^2\alpha = 0. \quad [3.10]$$

There exist two pure imaginary roots when $\beta_1 = 0$. The product of the coefficients of ω^2 and ω equals the constant term. Comparing with [3.7] above we get:

$$(k+\gamma_{\|}+\gamma_{\|})\gamma_{\|}(k+\gamma_{\|}+\gamma\lambda_0) = 2k\gamma\gamma_{\|}\lambda_0, \quad [3.11]$$

so that the bifurcation value is

$$\lambda_0 = \frac{(k+\gamma)(k+\gamma+\gamma_{\|})}{\gamma(k-\gamma-\gamma_{\|})}. \quad [3.12]$$

For $k > \gamma_{\|} + \gamma_{\|}$, and $\lambda < \lambda_0$, all roots have a negative real part, i.e., the state ($E = P = D = 1$) is asymptotically stable. This state is unstable if $\lambda > \lambda_0$, $k > \gamma + \gamma_{\|}$. Thus in order to get unstable solutions for the single mode laser the losses of the polarization and the inversion (bad cavity limit). In the good cavity case $k < \gamma + \gamma_{\|}$, the one-mode solution is always stable.

In order to evaluate $\beta_1'(\lambda_0)$, equate coefficients of same powers of ω in [3.7] and [3.10]:

$$-(k+\gamma+\gamma_{\|}) = 2\beta_1 + \alpha$$

$$\gamma_{\|}(k+\gamma+\lambda) = |\beta|^2 + 2\beta_1\alpha$$

$$-2k\gamma\gamma_{\|}\lambda = |\beta|^2\alpha. \quad [3.13]$$

Thus

$$\alpha = -(k+\gamma+\gamma_{\|}+2\beta_1), \quad [3.14]$$

and

$$\gamma_{\|}(k+\gamma+\gamma\lambda)\alpha = -2k\gamma\gamma_{\|}\lambda + 2\beta_1\alpha^2, \quad [3.15]$$

so that

$$-\gamma_{\|}(k+\gamma+\gamma\lambda)(k+\gamma+\gamma_{\|}+2\beta_1) = -2k\gamma\gamma_{\|}\lambda + 2\beta_1(k+\gamma+\gamma_{\|}+2\beta_1)^2. \quad [3.16]$$

Differentiating with respect to λ and recalling that $\beta_1(\lambda_0) = 0$, we obtain

$$\frac{d\beta_1(\lambda_0)}{d\lambda} = \frac{\gamma\gamma_{\|}(k-\gamma-\gamma_{\|})}{2\left[(k+\gamma+\gamma_{\|})^2 + \gamma_{\|}(k+\gamma+\gamma_{\|}\lambda_0)\right]}, \quad [3.17]$$

and we see that $\beta_1'(\lambda_0) > 0$ if $k > \gamma + \gamma_{\|}$, and a Hopf bifurcation takes place at $\lambda = \lambda_0$.

The frequency Ω at which the bifurcation occurs is given by $\beta_2^2 = |\beta|^2(\lambda_0)$ (when $\beta_1(\lambda_0) = 0$):

$$|\beta|^2(\lambda_0) = \frac{-2k\gamma\gamma_{\|}\lambda_0}{\alpha(\lambda_0)}, \quad [3.18]$$

then

$$\Omega^2 = |\beta|^2 = \frac{2k\gamma_{\|}(k+\gamma)}{k-\gamma-\gamma_{\|}}. \quad [3.19]$$

3.3. Stability of the bifurcating periodic solution

For simplicity we make $\gamma \equiv \gamma_{\parallel}$ in what follows. Then

$$\lambda_0 = \frac{(k+\gamma)(k+2\gamma)}{\gamma(k-2\gamma)}, \quad \Omega^2 = |\beta|^2 = \frac{2k\gamma(k+\gamma)}{k-2\gamma}. \qquad [3.20]$$

The equations for the deviations $z(t) = \mathrm{col}(\Delta E, \Delta P, \Delta D)$ are

$$\frac{dz(t)}{dt} = Lz(t) + N(t), \qquad [3.21]$$

where L is given in [3.6] and the nonlinear terms are

$$N_i = \frac{1}{2}\frac{\partial f_i}{\partial x_i \partial x_j}\bigg|_{E=P=D=1} z_i z_j. \qquad [3.22]$$

We look for a periodic solution in the form [2.30]. Then we get:

$$\left(\Omega\frac{\partial}{\partial\tau} - L_0\right)y(\tau,0) = 0, \qquad [3.23]$$

$$\left(\Omega\frac{\partial}{\partial\tau} - L_0\right)y(\tau,n) = \sum_{m=0}^{\infty} L_{n-m}(\lambda_n)y(\tau,m) - \omega_n\frac{\partial y(\tau,0)}{\partial\tau} + \varepsilon N, \qquad [3.24]$$

where $L_0 = L(\lambda_0)$. The solution of [3.23] is

$$y(\tau,0) = e^{i\tau}u + e^{-i\tau}u^*. \qquad [3.25]$$

where u is the eigenvector of L_0 with eigenvalue $i\Omega$:

$$L_0 u = i\Omega u. \qquad [3.26]$$

The adjoint matrix M_0 has the eigenvector ν:

$$M_0\nu = i\Omega\nu. \qquad [3.27]$$

A calculation gives

$$u = C\,\mathrm{col}\left[1, 1+\frac{i\Omega}{k}, \frac{i\Omega(k+\gamma)-\Omega^2}{k\gamma}\right], \qquad [3.28]$$

$$\nu = C\,\mathrm{col}\left[\frac{2i\Omega}{k} - 1, 1+\frac{i\Omega}{\gamma}, 1\right], \qquad [3.29]$$

where the constant C has been chosen to ensure the orthonormality relations

$$\langle u|\nu\rangle = 0 \quad , \quad \langle u|\nu^*\rangle = 1. \qquad [3.30]$$

We get

$$C^2 = -\frac{k\gamma}{2\Omega}\frac{\Omega + i(k+2\gamma)}{\Omega^2 + (k+2\gamma)^2}. \qquad [3.31]$$

To first order in ε, equation [3.24] yields

$$\left(\Omega\frac{\partial}{\partial\tau} - L_0\right)y(\tau,1) = L_1 y(\tau,0) - \omega_1\frac{\partial y(\tau,0)}{\partial\tau} + N(\varepsilon^0). \qquad [3.33]$$

so that

$$\varphi_1(\tau) = L_1 y(\tau,0) - \omega_1\frac{\partial y(\tau,0)}{\partial\tau} + N(\varepsilon^0), \qquad [3.34]$$

where

$$L_1 y(\tau,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma\lambda_1 & -\gamma\lambda_1 & 0 \end{pmatrix} y(t,0) = \left\{-\gamma\frac{\lambda_1}{k}e^{i\tau}C(2k+i\Omega) - \gamma\frac{\lambda_1}{k}e^{-i\tau}C^*(2k-i\Omega)\right\}\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad [3.35]$$

$$\omega_1\frac{\partial y(\tau,0)}{\partial\tau} = i\omega_1(e^{i\tau}u - e^{-i\tau}u^*), \qquad [3.36]$$

$$N(\varepsilon^0) = \begin{pmatrix} 0 \\ -\frac{2|C|^2\Omega^2}{k} + \frac{C^2\Omega}{k} e^{2i\tau}\left[i(k+\gamma)-\Omega\right] - \frac{C^{*2}\Omega}{k} e^{-2i\tau}\left[\Omega + i(k+\gamma)\right] \\ -2\lambda_0 \frac{|C|^2}{k} - \lambda_0 \frac{C^2}{k} e^{2i\tau}(i\Omega+k) - \lambda_0 \frac{C^{*2}\Omega}{k} e^{-2i\tau}(k-i\Omega) \end{pmatrix}, \qquad [3.37]$$

where we have used the notation $y_1(\tau,n)$ to indicate the i-component of the vector $y(\tau,n)$.

Then

$$\varphi_1(\tau)\tilde{\eta}_1 = -\frac{\gamma\lambda_1}{k} C^2(i\Omega+2k) - \frac{\gamma\lambda_1}{k} e^{-2i\tau}|C|^2 (2k-i\Omega) - i\omega_1 e^{-2i\tau}, \qquad [3.38]$$

and the solvability condition $[\varphi_1]_1 = (\varphi_1|\tilde{\eta}_1) = 0$, gives

$$(\varphi_1|\tilde{\eta}_1) = \frac{1}{2\pi}\int_0^{2\pi} \langle \varphi_1(\tau)|\tilde{\eta}_1\rangle d\tau = -\frac{\gamma}{k}\lambda_1 C^2(i\Omega+2k) = 0$$

$$\Rightarrow \lambda_1 = 0 \qquad [3.39]$$

and

$$\langle \varphi_1(\tau)|\tilde{\eta}_2\rangle = e^{i\tau}\langle \varphi_1(\tau)|\nu\rangle =$$

$$= -\frac{\gamma\lambda_1}{k}|C|^2 e^{2i\tau}(i\Omega+2k) - \frac{\gamma\lambda_1}{k}(2k-i\Omega)C^{*2} + i\omega_1 \qquad [3.40]$$

so that

$$\langle \varphi_1|\tilde{\eta}_2\rangle = -\frac{\gamma\lambda_1}{k}(2k-i\Omega) + i\omega_1 = 0$$

$$\Rightarrow \omega_1 = 0. \qquad [3.41]$$

Then equation [3.33] for $y(\tau,1)$ becomes

$$\left(\Omega\frac{\partial}{\partial\tau} - L_0\right) y(\tau,1) = N(\varepsilon). \qquad [3.42]$$

We need a particular solution for this equation in order to go to second order. The form of $N(\varepsilon)$ suggests

$$y(\tau,1) = Fe^{2i\tau} + Ge^{-2i\tau} + H. \qquad [3.43]$$

Substitution in [3.42] and comparison of coefficients leads to

$$F_1 G_1 = 0 \quad , \quad F_2 = G_2 = 0, \qquad [3.44]$$

$$F_3 G_3^* = \frac{C^2\Omega}{k\gamma}\left[i(k+\Omega)-\Omega\right], \qquad [3.45]$$

$$H_1 = H_2 = -\frac{|C|^2}{k\gamma}\left[1+\frac{2k\gamma^2}{k+2\gamma}\right],\ H_3 = \frac{2\Omega^2|C|^2}{k\gamma}, \qquad [3.46]$$

so that

$$y(\tau,1) = Fe^{2i\tau} + F^* e^{-2i\tau} + H. \qquad [3.47]$$

At second order, equation [3.24] gives

$$\left(\Omega\frac{\partial}{\partial\tau} - L_0\right) y(\tau,2) = L_2 y(\tau,0) + L_1 y(\tau,1) - \omega_2 \frac{\partial y(\tau,0)}{\partial\tau} + N(\varepsilon), \qquad [3.48]$$

where $L_i y(\lambda_1) = 0$. Then

$$L_2 y(\tau,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma\lambda_2 & -\gamma\lambda_2 & 0 \end{pmatrix} y(t,0) =$$

$$= -\left[\frac{C\gamma\lambda_2}{k}(2k+i\Omega)e^{i\tau} + \frac{C^*\gamma\lambda_2}{k}(2k+i\Omega)e^{-i\tau}\right]\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad [3.49]$$

$$\omega_2 \frac{\partial y(\tau,0)}{\partial \tau} = i\omega_2 (e^{i\tau} u - e^{-i\tau} u^*)\,, \qquad [3.50]$$

$$N(\varepsilon) = \gamma \begin{pmatrix} 0 \\ y_1(\tau,0) y_3(\tau,1) + y_1(\tau,1) y_3(\tau,0) \\ -\lambda_0 \left[y_1(\tau,0) y_2(\tau,1) + y_1(\tau,1) y_2(\tau,0) \right] \end{pmatrix}. \qquad [3.51]$$

Then

$$\varphi_2(t) = L_2 y(\tau,0) - \omega_2 \frac{\partial y(\tau,0)}{\partial \tau} + N(\varepsilon)\,. \qquad [3.52]$$

Inserted in the solvability condition $[\varphi_2]_1 = (\varphi_2 | \tilde{\eta}_1) = 0$, gives

$$-C\gamma\lambda_2 \frac{(i\Omega + 2k)}{k} + H_3(\gamma + i\Omega) - i\omega_2 + C^* F_3(\gamma + i\Omega) + C\Omega H_1 \frac{[i(k+\Omega) - \Omega]}{k\gamma}(\gamma + i\Omega) - \lambda_0 \gamma C H_1 \frac{(i\Omega + k)\Omega}{k} = 0\,. \qquad [3.53]$$

Substitution of the expressions for F_3 and C and taking the real part, we find after some algebra:

$$\lambda_2(k,\gamma) = \frac{1}{\gamma^2 (2ka - b\Omega)} \Big\{ \gamma c_3 (a\gamma - b\Omega) - \lambda_0 c_1 \gamma^2 (ak - b\Omega) - c_1 \Omega \big[\Omega(a\gamma + b\Omega) + (k+\gamma)(a\Omega + b\gamma) \big] + \Omega(a\Omega - b\gamma)\big[2b\Omega - (k+\Omega)(a^2 - b^2)\big] + (a\gamma - b\Omega)[2b(k+\gamma) + \Omega(a^2 - b^2)] \Big\}\,, \qquad [3.54]$$

where

$$a = \left[\frac{k\gamma}{4\Omega} \frac{1}{(\Omega^2 + k + 2\gamma)} \right]^{1/2} \left\{ \left[\Omega^2 + (k+2\gamma)^2 \right]^{1/2} - \Omega \right\}^{1/2}, \qquad [3.55]$$

$$b = -\left[\frac{k\gamma}{4\Omega} \frac{1}{(\Omega^2 + k + 2\gamma)} \right]^{1/2} \frac{k + 2\gamma}{\left\{ \left[\Omega^2 + (k+2\gamma)^2 \right]^{1/2} - \Omega \right\}^{1/2}}\,,$$

$$c_3 = \frac{\Omega}{(\Omega^2 + k + 2\gamma)} \frac{\left\{ \Omega^2 + (k+2\gamma)^2 - \Omega \left[\Omega^2 + (k+2\gamma)^2 \right]^{1/2} \right\}}{\left[\Omega^2 + (k+2\gamma)^2 \right]^{1/2}}$$

$$c_1 = -\frac{1}{2\Omega^2} \left[1 + \frac{2k\gamma^2}{k + 2\gamma} \right] c_3$$

$$\Omega^2 = \frac{2k\gamma(k+\gamma)}{k - 2\gamma}$$

$$\lambda_0 = \frac{(k+\gamma)(k+2\gamma)}{\gamma(k-2\gamma)}\,.$$

Equation [3.53] also gives ω_2 by taking its imaginary part. The coeffcient $\lambda_2(k,\gamma)$ becomes singular at the values $k = 0, \gamma = 0$; but in this case the single mode laser equations are trivial.

Figure 1 shows the critical boundary $\lambda_2(k,\gamma) = 0$ of stability in the space of parameters (k,γ) for the periodic orbit of the laser equations [3.1] undergoing a Hopf bifurcation at the value $\lambda = \lambda_0$. This boundary separates the region where this periodic orbit is unstable $(\lambda_2(k,\gamma) < 0)$ from the region where this orbit is stable $(\lambda_2(k,\gamma) > 0)$ on the parameter plane (k,γ).

4. Conclusions

The perturbation method of Section 3 for analytically determining the stability of bifurcating periodic orbits is simple in principle but may become an arduous task in actual problems. However the explicit formulas

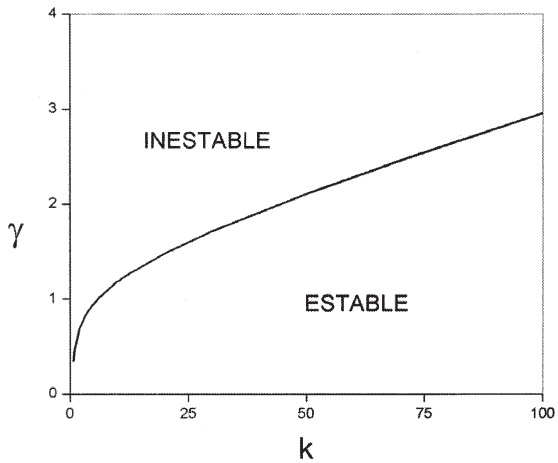

Figure 1. Critical boundary $\lambda_2(k,\gamma)=0$ of stability in the space of parameters (k,γ) for the bifurcating periodic orbit of the single mode laser equations [3.1]. This boundary separates the domain where this periodic orbit is unstable $(\lambda_2(k,\gamma)<0)$ from the domain where this orbit is stable $(\lambda_2(k,\gamma)>0)$ on the parameter plane (k,γ) as indicated.

for the coefficient of stability λ_2 could help to discover the regions of stable oscillations in the parameter space and to predict the onset of instabilities and chaotic behavior.

Numerical work has shown wide regions of instabilities in the Lorenz system (15, 16). However some intervals of stable periodic solutions have been found. Thus we would expect some regions of oscillatory stability in the single mode laser. The region of instability $(\lambda_2(k,\gamma)<0)$ predicted by Equation [3.56] agrees qualitatively with comparable points of the Lorenz systems. The critical curve $\lambda_2(k,\gamma)=0$ separates the domains of stable oscillations of the single mode laser ($\lambda_2>0$) from the unstable ones ($\lambda_2<0$) on the space of parameters (k,γ).

Mandel and Zeghlache (12) have obtained analytic phase diagrams of the stability coefficient of a detuned mode laser by using a different method of stability analysis. Lugiato *et al.* (17) have numerically discovered periodic behavior in a single-mode mean-field model of optical bistability in a neighborhood of the upper branch of the steady state curve. The system of equations involved in that case are slightly more complicated than [3.1] and it might be valuable to determine the stability of the oscillations analytically by using the method proposed in this article.

The influences of external noise on oscillations have been investigated for some systems (18). One advantage of our analytical approach is the possibility o conceptual generalizations, as to include the effect of external noise on the stability of bifurcating periodic orbits. A further extension of the method presented here could use the stability analysis of Hopf bifurcations to introduce a parameter control feedback on periodic orbits in dynamical systems (19).

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Appendix I: Floquet exponents and stability

Let $z(t)$ be a periodic solution of [2.5] obtained according to the procedure given in Section 2 and form the linearized equations for perturbations h. We get:

$$\omega(\varepsilon)\frac{\partial h}{\partial \tau} - L(\lambda(\varepsilon))h = \left[N(z(t),\varepsilon)\right] \equiv F(\tau,\varepsilon). \quad [I.1]$$

We seek solutions of [I.1] in the form

$$h = e^{-\sigma\tau}\Gamma(\tau)\,, \qquad [I.2]$$

where $\sigma = \sigma(\lambda)$ is the so-called Floquet exponent, and $\Gamma(\tau) = \Gamma(\tau + 2\pi)$.

Substitution of [I.2] in [I.1] yields

$$-\sigma\omega(\varepsilon)\Gamma + \omega(\varepsilon)\frac{\partial\Gamma}{\partial\tau} - L(\varepsilon)\Gamma = F(\tau)\,. \qquad [I.3]$$

Assume

$$\Gamma(\tau,\varepsilon) = \alpha(\varepsilon)\frac{\partial h}{\partial\tau} + \gamma(\tau,\varepsilon).[I.4]$$

Substitution in [I.3] gives

$$\omega\frac{\partial\gamma}{\partial\tau} - L(\varepsilon)\gamma = \sigma\omega\gamma + \sigma\omega a\frac{\partial h}{\partial\tau} + F(\tau,\varepsilon)\,. \qquad [I.5]$$

As in Section 2, we look for a solution in series

$$\begin{pmatrix}\gamma(\tau,\varepsilon)\\ \sigma(\varepsilon)\\ a(\varepsilon)\end{pmatrix} = \sum_{n=0}^{\infty}\varepsilon^n\begin{pmatrix}\gamma_n(\tau)\\ \sigma_n\\ a_n\end{pmatrix},[I.6]$$

and require, as in the construction of the periodic solutions, that

$$[\gamma_0]_1 = 1, \quad [\gamma_n]_1 = 0 \quad ; \quad (n>0). \qquad [I.7]$$

To lowest order, substitution of [I.6] in equation [I.5] and conditions [I.7] lead to

$$-\sigma_0\omega_0(ia_0+1) = 0\,, \qquad [I.8]$$

so that $\sigma_0 = 0$ and therefore $\gamma_0 = h$. At first order, and using the solvability conditions [I.7] in this case gives $\sigma_1 = 0$. At second order, substitution of [I.6] in equation [I.5] gives

$$\left(\omega_0\frac{\partial}{\partial\tau} - L_0\right)\gamma_2 = \sigma_2\omega_0\left(a_0\frac{\partial h}{\partial\tau} + h\right) + \omega_2\frac{\partial h}{\partial\tau} + F(\tau,2). \qquad [I.9]$$

Conditions [I.7] give

$$-\sigma_2\omega_0(ia_0+1) + i\omega_2 + [F(\tau,2)]_1 = 0. \qquad [I.10]$$

Taking real and imaginary parts and using [2.37] and [2.38] we get

$$\sigma_2 = -\frac{2\lambda_2\,\mathrm{Re}\,\xi_2}{\omega_0}. \qquad [I.11]$$

Thus to second order, the Floquet exponent is

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 = \varepsilon^2\sigma_2. \qquad [I.12]$$

Recall from [I.2] that solutions of the linearized equation [I.1] decay when $\sigma > 0$ and that $\mathrm{Re}\,\xi_2 > 0$ from [2.38]. Thus we conclude that subcritical periodic motions ($\lambda_2 < 0$) are unstable ($\sigma_2 > 0$) and supercritical periodic motions are stable ($\sigma_2 < 0$) in the linearized theory.

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