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Fifth order linear recurrence sequences and their positivity

Secuencias recurrentes lineales de quinto orden y su positividad

Vichian Laohakosol (fscivil@ku.ac.th)
Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900 Thailand

Pinthira Tangsupphathawat (t.pinthira@hotmail.com)
Department of Mathematics, Faculty of Science and Technology, Phranakhon Rajabhat University, Bangkok 10220, Thailand

Abstract

Consider a fifth order linear recurrence sequence with integer coefficients. The problem whether each of its elements is nonnegative is shown to be decidable.

Key words and phrases: Positivity problem, recurrence sequences, decidability.

Resumen

Considere una secuencia de recurrencia lineal de quinto orden con coeficientes enteros. Se muestra que el problema de si cada uno de sus elementos es no negativo es decidible.

Palabras y frases clave: Problema de positividad, secuencia recurrente, decidibilidad.

1 Introduction

A fifth order linear recurrence sequence considered here is a sequence of integers $(u_n)_{n\geq 0}$ satisfying

$$u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3} + a_4u_{n-4} + a_5u_{n-5} \quad (n \geq 5), \tag{1}$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z} - \{0\}$. As elaborated in [10], there are three main decision problems related to linear recurrence sequences, namely,

- **The Skolem Problem:** Does a given linear recurrence sequence have a zero?
- **The Positivity Problem:** Are all the terms of a given linear recurrence sequence positive? This problem has two variations whether strict or non-strict positivity is required; here we are only interested in the non-strict case.

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Corresponding author: Vichian Laohakosol

- **The Ultimate Positivity Problem:** Is a given linear recurrence sequence ultimately positive, i.e., are all the terms of the sequence positive except possibly for a finite number of exceptions?

It is known ([10, Section 4], [5]) that the decidability of the positivity problem implies the decidability of the Skolem problem.

The *positivity problem* associated with (1) asks whether it is possible to decide whether $u_n \geq 0$ for all $n \geq 0$? At present, this problem remains open. However, certain partial results have already appeared. The positivity problem for sequences satisfying a second order linear recurrence relation was shown to be decidable by Halava-Harju-Hirvensalo in 2006, see [6]; see also [2] and [3]. The positivity problem for sequences satisfying a third or a fourth order linear recurrence relation was shown to be decidable in [7], [8] and [11]. We show here that the same conclusion holds for a fifth order linear recurrence sequence.

Theorem 1.1. *The positivity problem is decidable for each sequence of integers satisfying a fifth order linear recurrence with integer coefficients.*

Our analysis here treats all possible shapes of sequence elements similar to the fourth order case in [11]. Apart from many more cases to be scrutinized and tools employed in the fourth order case, there are certain noteworthy ingredients needed, namely:

- A result about spanning set of a $\mathbb{Z}$-module generated by finitely many (irrational) real numbers [2, Lemma 3];
- The multi-dimensional Kronecker-Weyl theorem [4, Theorems IV.I and IV.II], [2, Theorem 4];
- A result about values of a linear combination of cosine functions whose arguments are rational multiples of π [2, Lemma 7];
- A deep result about linear forms in logarithm [1].

2 Preliminaries

Our first auxiliary result gives an explicit shape of elements of a linear recurrence sequence with integer coefficients. The general characteristic polynomial associated with the recurrence relation (1) of order k is

$$\mathrm{Char}(z) := z^k - a_1 z^{k-1} - \cdots - a_{k-1} z - a_k.$$

Let all the distinct (nonzero) real roots of $\mathrm{Char}(z)$ be λ_m with multiplicities $\mu_m + 1$ where $\mu_m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $m = 1, \ldots, r$, and let all the distinct (nonzero) complex conjugate pairs of the roots be $\gamma_j \left(= |\gamma_j| e^{i\theta_j}\right), \bar{\gamma}_j$ with multiplicities $\nu_j + 1$ ($\nu_j \in \mathbb{N}_0, j = 1, \ldots, s$), so that

$$\mu_1 + \cdots + \mu_r + 2\nu_1 + \cdots + 2\nu_s + r + 2s = k.$$

It is well-known [12] that each sequence element satisfying (1) has the form

$$u_n = \sum_{m=1}^{r} P_m(n)\lambda_m^n + \sum_{j=1}^{s} \left\{ E_j(n)\gamma_j^n + F_j(n)\bar{\gamma}_j^n \right\} \quad (n \in \mathbb{N}_0),$$

where

$$
\begin{aligned}
P_m(n) &:= \sum_{a=0}^{\mu_m} A_{m,a}\, n^a \quad (A_{m,\mu_m} \neq 0; m = 1, 2, \dots, r) \\
E_j(n) &:= \sum_{b=0}^{\nu_j} B_{j,b}\, n^b;\ F_i(n) := \sum_{b=0}^{\nu_j} C_{j,b}\, n^b\ (B_{j,\nu_j} \neq 0, C_{j,\nu_j} \neq 0, j = 1, \dots, s).
\end{aligned} \tag{2}
$$

We claim that the coefficients $A_{m,a}$ and those of $P_m(n)$ are real, while those of $E_j(n)$ and $F_j(n)$ are complex conjugate pairs, i.e., $\overline{B}_{j,b} = C_{j,b}$. To see this, note that from the given initial real values $u_0, u_1, \dots, u_{k-1}$, solving for the coefficients via Cramer's rule we get, e.g. for the first coefficient of $P_1(n)$,

$$
\det \begin{pmatrix} X & Y & \overline{Y} \end{pmatrix} A_{1,0} = \det \begin{pmatrix} X(u) & Y & \overline{Y} \end{pmatrix} \tag{3}
$$

where

$$
X = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
\lambda_1 & \cdots & \lambda_1 & \cdots & \lambda_r & \cdots & \lambda_r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_1^{k-1} & \cdots & (k-1)^{\mu_1}\lambda_1^{k-1} & \cdots & \lambda_r^{k-1} & \cdots & (k-1)^{\mu_1}\lambda_r^{k-1}
\end{bmatrix},
$$

$$
Y = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
\gamma_1 & \cdots & \gamma_1 & \cdots & \gamma_s & \cdots & \gamma_s \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_1^{k-1} & \cdots & (k-1)^{\nu_1}\gamma_1^{k-1} & \cdots & \gamma_s^{k-1} & \cdots & (k-1)^{\nu_1}\gamma_s^{k-1}
\end{bmatrix},
$$

the matrix $\overline{Y}$ is obtained from Y by replacing each of its elements with its complex conjugate, and $X(u)$ is the matrix obtained from X by replacing the first column by the column vector $\begin{bmatrix} u_0 & u_1 & \dots & u_{k-1} \end{bmatrix}^t$. Taking complex conjugate on both sides of (1.3), we get

$$
\det \begin{pmatrix} X & \overline{Y} & Y \end{pmatrix} \overline{A}_{1,0} = \det \begin{pmatrix} X(u) & \overline{Y} & Y \end{pmatrix}. \tag{4}
$$

Employing elementary operations on the determinants of both sides in this last equation, we obtain (3) with $\overline{A}_{1,0}$ in place of $A_{1,0}$, which shows that $A_{1,0} \in \mathbb{R}$. Similar arguments applied to other coefficients validate the claim.

Returning to our case of fifth order, the characteristic polynomial associated with the relation (1) is

$$
\mathrm{Char}(z) := z^5 - a_1 z^4 - \cdots - a_4 z - a_5.
$$

Let $\lambda_k \in \mathbb{C} \setminus \{0\}$ $(k = 1, \dots, m)$ be all the distinct roots of $\mathrm{Char}(z)$, with multiplicities $\ell_1, \dots, \ell_m$ respectively, so that $\ell_1 + \cdots + \ell_m = 5$. Each sequence element satisfying (1) can be written as

$$
u_n = \sum_{k=1}^{m} P_k(n)\lambda_k^n, \qquad (n \geq 0),
$$

with $P_k(n) \in \mathbb{C}[n]$, $\deg P_k = \ell_k - 1$ $(k = 1, \dots, m)$. The roots of $\mathrm{Char}(z)$ having the largest absolute value are called *dominating roots*. The following result of Bell-Gerhold, [2, Theorem 2], helps reducing about half the number of cases to be considered.

Proposition 2.1. *Let (u_n) be a nonzero recurrence sequence with no positive dominating characteristic root, then the sets $\{n \in \mathbb{N} : u_n > 0\}$ and $\{n \in \mathbb{N} : u_n < 0\}$ have positive density, and so both sets contain infinitely many elements.*

Apart from those in [11, Lemma 2.3], some more forms of sequence elements which have already been shown to be decidable are contained in the next lemma.

Lemma 2.1. *The positivity problem for the following forms of sequence elements are decidable.*

1. $u_n = A\lambda_1^n + B\lambda_2^n \quad (A, B \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
2. $u_n = (A + B)\lambda^n \quad (A, B \in \mathbb{R};\ \lambda \in \mathbb{R} \setminus \{0\})$
3. $u_n = A\lambda^n + \bar{A}\bar{\lambda}^n \quad (A \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \{0\})$
4. $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
5. $u_n = (A + Bn + Cn^2)\lambda^n \quad (A, B, C \in \mathbb{R};\ \lambda \in \mathbb{R} \setminus \{0\})$
6. $u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
7. $u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
8. $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n \quad (A, B, C, D \in \mathbb{R};\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
9. $u_n = (A + Bn + Cn^2 + Dn^3)\lambda^n \quad (A, B, C, D \in \mathbb{R};\ \lambda \in \mathbb{R} \setminus \{0\})$
10. $u_n = A\lambda_1^n + (B + Cn + Dn^2)\lambda_2^n \quad (A, B, C, D \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
11. $u_n = (A + Bn)\lambda_1^n + (C + Dn)\lambda_2^n \quad (A, B, C, D \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
12. $u_n = A\lambda_1^n + B\lambda_2^n + (C + Dn)\lambda_3^n \quad (A, B, C, D \in \mathbb{R};\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$ *distinct*$)$
13. $u_n = (A + Bn)\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n \quad (A, B \in \mathbb{R};\ \lambda_1 \in \mathbb{R} \setminus \{0\};\ C \in \mathbb{C};\ \lambda_2 \in \mathbb{C} \setminus \mathbb{R})$
14. $u_n = (A + Bn)\lambda^n + (\bar{A} + \bar{B}n)\bar{\lambda}^n \quad (A, B \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \mathbb{R})$
15. $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (A, B \in \mathbb{C};\ \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ *distinct*$)$
16. $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (A, B \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ *distinct*$;\ C \in \mathbb{C};\ \lambda_3 \in \mathbb{C} \setminus \mathbb{R})$.

3 Proof of Theorem 1.1

In this section the proof of Theorem 3.1 is exposed. This is divided into several cases taking into account the nature of the roots of $\mathrm{Char}(z)$.

3.1 Char(z) has only real roots

In this case, the general term of the sequence is

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0,\ m \leq 5).$$

This case is decidable by the same proof as that in [11, Section 3].

3.2 Char(z) has non-real roots

The possible shapes of the five roots are:

I. One real number and two identical complex conjugate pairs, denoted by $C(rz^2\bar{z}^2)$;

II. Three identical real numbers and one complex conjugate pair, denoted by $C(r^3z\bar{z})$;

III. Two distinct real numbers one of whose is of multiplicity 2 and one complex conjugate pair, denoted by $C(r_1r_2^2z\bar{z})$;

IV. There distinct real numbers and one complex conjugate pair, denoted by $C(r_1r_2r_3z\bar{z})$.

V. One real number and two complex conjugate pairs, denoted by $C(rz_1\bar{z}_1z_2\bar{z}_2)$;

3.2.1 Case I: C$(rz^2\bar{z}^2)$

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n + (\bar{B} + \bar{C}n)\bar{\lambda}_2^n \quad (n \geq 0),$$

where $A, \lambda_1 \in \mathbb{R} \setminus \{0\}$; $B, C \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_2 = |\lambda_2|e^{i\theta}$, $B = |B|e^{i\varphi_1}$ and $C = |C|e^{i\varphi_2}$ where $\theta, \varphi_1, \varphi_2 \in [-\pi, \pi]$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + 2|\lambda_2|^n\{|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta)\}.$$

By Proposition 2.1, we need only treat the case where there is a positive dominating root, i.e., $\lambda_1 \geq |\lambda_2| > 0$. There are two subcases.

Subcase I.1 $\lambda_1 = |\lambda_2|$**.**
Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + 2|B|\cos(\varphi_1 + n\theta) + 2n|C|\cos(\varphi_2 + n\theta)\}.$$

Then, the sequence (u_n) is nonnegative if and only if, for all n,

$$A + 2|B|\cos(\varphi_1 + n\theta) + 2n|C|\cos(\varphi_2 + n\theta) \geq 0. \tag{5}$$

Since

$$\operatorname{sign}\left(A + 2|B|\cos(\varphi_1 + n\theta) + 2n|C|\cos(\varphi_2 + n\theta)\right) = \operatorname{sign}\left(\cos(\varphi_2 + n\theta)\right),$$

when n is large enough, provided $|C|\cos(\varphi_2 + n\theta) \neq 0$. By [11, Lemma 2.2 III], $\cos(\varphi_2 + n\theta)$ takes some positive and some negative values for infinitely many $n \in \mathbb{N}$. Thus, (5) holds only when $C = 0$, which reduces the shape of u_n to the form *7.*, Lemma 2.1, and so it is decidable.

Subcase I.2: $\lambda_1 > |\lambda_2|$**.**
Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + 2(|\lambda_2|/\lambda_1)^n(|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta))\}.$$

The sequence (u_n) is nonnegative if and only if

$$A + 2(|\lambda_2|/\lambda_1)^n(|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta)) \geq 0 \quad (n \geq 0).$$

- If $A = 0$, the shape of u_n reduces to the form *11.*, Lemma 2.1.
- If $A < 0$, then $A + 2(|\lambda_2|/\lambda_1)^n(|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta)) < 0$ for all sufficiently large n, and so this case is untenable.
- If $A > 0$, since

$$A + 2(|\lambda_2|/\lambda_1)^n(|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta)) \to A > 0 \ (n \to \infty),$$

there is an explicitly computable least $M_1 \in \mathbb{N}_0$, depending on $A, B, C, \lambda_1, \lambda_2, \theta_1, \theta_2, \varphi_1, \varphi_2$ for which

$$A + 2(|\lambda_2|/\lambda_1)^n(|B|\cos(\varphi_1 + n\theta) + n|C|\cos(\varphi_2 + n\theta)) \geq 0 \quad \text{for all } n \geq M_1.$$

Thus, the sequence (u_n) is nonnegative if and only if $M_1 = 0$.

3.2.2 Case II: C($r^3 z\bar{z}$)

In this case, the general term of the sequence is $u_n = (A + Bn + Cn^2)\lambda_1^n + D\lambda_2^n + \bar{D}\bar{\lambda}_2^n$. This case is decidable by the same proof as in Case 4.1 of [11, p. 137].

3.2.3 Case III: C($r_1 r_2^2 z\bar{z}$)

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n + D\lambda_3^n + \bar{D}\bar{\lambda}_3^n \qquad (n \geq 0),$$

where $A, B, C \in \mathbb{R}; \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$; $D \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_3 = |\lambda_3|e^{i\theta}$ and $D = |D|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi]$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n + 2|D||\lambda_3|^n\cos(\varphi + n\theta).$$

We assume that the three coefficients A, C, D are all nonzero, for otherwise we would end in the cases *13.* or *16.*, or the case *6.*, Lemma 2.1. We treat two subcases depending on whether $|\lambda_1| = |\lambda_2|$.

Subcase III.1: $|\lambda_1| = |\lambda_2|$*, and so* $\lambda_2 = -\lambda_1$

If $\lambda_1 > 0$, then the sequence term is of the form $u_n = \{A + (-1)^n(B + Cn)\}\lambda_1^n + D\lambda_3^n + \bar{D}\bar{\lambda}_3^n$, i.e.,

$$\begin{aligned} u_{2k} &= \{A + B + 2Ck\}\lambda_2^{2k} + D\lambda_3^{2k} + \bar{D}\bar{\lambda}_3^{2k} \\ u_{2k+1} &= \{A - B - C - 2Ck\}\lambda_2^{2k+1} + D\lambda_3^{2k+1} + \bar{D}\bar{\lambda}_3^{2k+1} \quad (k \geq 0). \end{aligned}$$

The two sequences (u_{2k}) and (u_{2k+1}) are decidable because they are of the form *13.*, Lemma 2.1.

If $\lambda_1 < 0$, then the sequence term is of the form $u_n = \{(-1)^n A + B + Cn\}\lambda_2^n + D\lambda_3^n + \bar{D}\bar{\lambda}_3^n$, i.e.,

$$\begin{aligned} u_{2k} &= \{A + B + 2Ck\}\lambda_2^{2k} + D\lambda_3^{2k} + \bar{D}\bar{\lambda}_3^{2k} \\ u_{2k+1} &= \{-A + B + C + 2Ck\}\lambda_2^{2k+1} + D\lambda_3^{2k+1} + \bar{D}\bar{\lambda}_3^{2k+1} \quad (k \geq 0). \end{aligned}$$

The two sequences (u_{2k}) and (u_{2k+1}) are decidable because they are of the form *13.*, Lemma 2.1.

Subcase III.2: $|\lambda_1| \neq |\lambda_2|$.

There are three possibilities for the absolute values of λ_i.

III.2.1: $|\lambda_1| = |\lambda_3|$.

By Proposition 2.1, we need only treat the case where there is a positive dominating root. We consider two possibilities corresponding to the positive dominating root being λ_1 or λ_2.

1. λ_1 *is the positive dominating root, and so* $\lambda_1 > |\lambda_2|$.

   Rewrite the general term of the sequence

   $$u_n = \lambda_1^n \{A + 2|D|\cos(\varphi + n\theta) + (B + Cn)(\lambda_2/\lambda_1)^n\} \quad (n \geq 0).$$

   Then, the sequence (u_n) is nonnegative if and only if, for all n,

   $$A \geq -2|D|\cos(\varphi + n\theta) - (B + Cn)(\lambda_2/\lambda_1)^n \quad (n \geq 0). \tag{6}$$

   - If θ is a rational multiple of π, say, $\theta = s\pi/t$ where $s, t \in \mathbb{Z}^+ \setminus \{0\}$ and $\gcd(s,t) = 1$. By Lemma 2.2 of [11], $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ distinct explicit (positive and negative) values at $n \in \{0, 1, \ldots, 2t-1\} (\text{mod } 2t)$. Let

     $$P := \{p_1, \ldots, p_s\}, \quad Q := \{q_1, \ldots, q_r\}$$

     be subsets of $\{0, 1, \ldots, 2t-1\}$ such that

     $$c_i = \cos(\varphi + p_i\theta) < 0 \qquad (1 \leq i \leq s), \qquad d_j = \cos(\varphi + q_j\theta) \geq 0 \quad (1 \leq j \leq r).$$

     Since

     $$\mathbb{N}_0 = (p_1 + 2t\mathbb{N}_0) \cup \cdots \cup (p_s + 2t\mathbb{N}_0) \cup (q_1 + 2t\mathbb{N}_0) \cup \cdots \cup (q_r + 2t\mathbb{N}_0),$$

     and for each $\ell \in \mathbb{N}$, let

     $$\begin{aligned} F_i(\ell) &= -2|D|c_i - (B + C(p_i + 2t\ell))(\lambda_2/\lambda_1)^{p_i + 2\ell t} \quad (i = 1, \ldots, s) \\ N_j(\ell) &= -2|D|d_j - (B + C(q_j + 2t\ell))(\lambda_2/\lambda_1)^{q_j + 2\ell t} \quad (j = 1, \ldots, r). \end{aligned}$$

     If $-(B + C(p_i + 2\ell t)(\lambda_2/\lambda_1))^{p_i + 2\ell t} \leq 0$ for all $\ell \in \mathbb{N}$, then we see clearly that

     $$\sup_{\ell \in \mathbb{N}_0} F_i(\ell) = -2|D|c_i.$$

     If there exists $\ell_0 \in \mathbb{N}$ such that

     $$-(B + C(p_i + 2\ell t)(\lambda_2/\lambda_1)^{p_i + 2\ell t} > 0,$$

     since $-(B + C(p_i + 2\ell t))(\lambda_2/\lambda_1)^{p_i + 2\ell t} \to 0$ $(\ell \to \infty)$, there is an $L_i \in \mathbb{N}_0$, $L_i \geq \ell_0$ for which

     $$\sup_{\ell \in \mathbb{N}_0} F_i(\ell) = -2|D|c_i + \max_{\ell \in \{0,1,\ldots,L_i\}} \left\{ -\left(B + C(p_i + 2\ell t)\right) (\lambda_2/\lambda_1)^{p_i + 2\ell t} \right\};$$

call this maximum M_i. Similarly, if $-(B+C(q_j+2\ell t)(\lambda_2/\lambda_1)^{q_j+2\ell t} \le 0$, for all $\ell \in \mathbb{N}$, then $\sup_{\ell\in\mathbb{N}_0} N_j(\ell) = -2|D|d_j$, while if there exists $\ell_1 \in \mathbb{N}$ such that

$$-(B+C(q_j+2\ell_1 t)(\lambda_2/\lambda_1)^{q_j+2\ell_1 t} > 0,$$

then $\max_{\ell\in\mathbb{N}_0} N_j(\ell)$ is attainable; call this maximum K_j. Thus, (6) holds if and only if

$$A \ge \max_{1\le i\le s, 1\le j\le r} \{-2|D|c_i, M_i, -2|D|d_j, K_j\},$$

and so this case is decidable.

- If θ_1 is not a rational multiple of π, rewrite the terms of the sequence as

$$u_n = |\lambda_2|^n \{(A+2|D|\cos(\varphi+n\theta))(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n\}.$$

The sequence $\{u_n\}$ is nonnegative if and only if

$$(A+2|D|\cos(\varphi+n\theta))(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n \ge 0 \quad (n\ge 0) \tag{7}$$

We consider four separate situations.

(a) If $A<0$, since the values of $\cos(\varphi+n\theta)$ is dense in the closed interval $[-1,1]$, there is a computable nonnegative integer N_A such that

$$(A+2|D|\cos(\varphi+N_A\theta))(\lambda_1/|\lambda_2|)^{N_A} + (B+CN_A)(\lambda_2/|\lambda_2|)^{N_A} < 0$$

and so (25) cannot be fulfilled.

(b) If $0<A<2|D|$, let $\Delta = 2|D| - A > 0$ so that

$$\begin{aligned}&(A+2|D|\cos(\varphi+n\theta))(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n \\ &= \{2|D|(1+\cos(\varphi+n\theta)) - \Delta)\}(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n.\end{aligned} \tag{8}$$

Taking a subsequence (n_k) for which $\cos(\varphi_1+n_k\theta_1) \to -1$ $(k\to\infty)$, the left-hand side of (8) tends to $-\infty$, showing that (7) cannot be fulfilled.

(c) If $A > 2|D| > 0$, let $\delta = A - 2|D| > 0$. Since

$$\begin{aligned}&(A+2|D|\cos(\varphi+n\theta))(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n \\ &\ge \delta(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n \to \infty \quad (n\to\infty),\end{aligned}$$

there is a computable nonnegative integer N^* such that

$$(A+2|D|\cos(\varphi+n\theta))(\lambda_1/|\lambda_2|)^n + (B+Cn)(\lambda_2/|\lambda_2|)^n \ge 0$$

for all $n \ge N^*$. Thus, (7) holds if and only if $N^* = 0$.

(d) If $A = 2|B|$, then (7) becomes

$$2|D|(1 + \cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n + (B + Cn)(\lambda_2/|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{9}$$

By [8, Lemma 2.2 II], there is $N_0 \in \mathbb{N}_0$ such that $1 + \cos(\varphi + N_0\theta) = 0$, which must be unique by [11, Claim 1, p. 140], then for (9) to hold we must have

$$(B + CN_0)(\lambda_2/|\lambda_2|)^{N_0} \geq 0.$$

Using arguments similar to [8, Lemma 2.2], we deduce that

$$2|D|\{1 + \cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + (B + Cn)(\lambda_2/|\lambda_2|)^n \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N}_0$, depending on $B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$2|D|\{1 + \cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + (B + Cn)(\lambda_2/|\lambda_2|)^n \geq 0 \ \text{ for all } n \geq N_1.$$

Using all the obtained information, we conclude that (25) holds if and only if $N_1 = 0$.

2. *λ_2 is the positive dominating root, and so $\lambda_2 > |\lambda_1| = |\lambda_3| > 0$.*

Rewrite the general term of the sequence

$$u_n = \lambda_2^n \{(A_n + 2|D|\cos(\varphi + n\theta))(|\lambda_1|/\lambda_2)^n + B + Cn\} \quad (n \geq 0),$$

where $A_n = (-1)^n A$ if $\lambda_1 < 0$, and $A_n = A$ if $\lambda_1 > 0$. The sequence (u_n) is nonnegative if and only if

$$\{A_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_1|/\lambda_2)^n + B + Cn \geq 0 \quad \text{for all} \quad n \geq 0. \tag{10}$$

We consider two possibilities.

- If $C < 0$, then

$$\{A_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_1|/\lambda_2)^n + B + Cn < 0$$

for all sufficiently large n, and so (10) is untenable.

- If $C > 0$, since

$$\{A_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_1|/\lambda_2)^n + B + Cn \to \infty \quad (n \to \infty),$$

there is an explicitly computable least $T_0 \in \mathbb{N}_0$, depending explicitly on $A, B, C, D, \varphi, \theta, \lambda_1, \lambda_2$ for which

$$\{A_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_1|/\lambda_2)^n + B + Cn \geq 0 \quad \text{for all} \quad n \geq T_0.$$

Thus, the sequence (u_n) is nonnegative if and only if

$$\{A_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_1|/\lambda_2)^n + B + Cn \geq 0 \ \text{ (for all } \ n \geq 0) \Leftrightarrow T_0 = 0.$$

III.2.2: $|\lambda_2| = |\lambda_3|$**.**

By Proposition 2.1, we need only treat the case where there is a positive dominating root. We consider two possibilities corresponding to the positive dominating root being λ_1 or λ_2.

1. λ_1 *is the positive dominating root, so that* $\lambda_1 > |\lambda_2| = |\lambda_3|$.

   Rewrite the general term of the sequence as

   $$u_n = \lambda_1^n \{A + (C_n + 2|D|\cos(\varphi + n\theta))(|\lambda_2|/\lambda_1)^n\} \quad (n \geq 0),$$

   where $C_n = (-1)^n(B + Cn)$ if $\lambda_2 < 0$ and $C_n = B + Cn$ if $\lambda_2 > 0$. The sequence (u_n) is nonnegative if and only if

   $$A + \{C_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_2|/\lambda_1)^n \geq 0 \quad (n \geq 0). \tag{11}$$

   Note that

   $$A + \{C_n + 2|D|\cos(\varphi + n\theta)\}(|\lambda_2|/\lambda_1)^n \to A \quad (n \to \infty). \tag{12}$$

   If $A < 0$, then (12) shows that (11) cannot be fulfilled. If $A > 0$, then (12) implies that there is an explicitly computable least integer $T_1 \in \mathbb{N}_0$, depending on $A, B, C, D, \varphi, \theta, \lambda_1, \lambda_2$ such that (11) holds for all $n \geq T_1$. Consequently, in this case the sequence (u_n) is nonnegative if and only if $T_1 = 0$.

2. λ_2 *is the positive dominating root, so that* $\lambda_2 = |\lambda_3| > |\lambda_1|$.

   Rewrite the general term of the sequence as

   $$u_n = \lambda_2^n \{A(\lambda_1/\lambda_2)^n + B + Cn + 2|D|\cos(\varphi + n\theta)\} \quad (n \geq 0).$$

   Then, the sequence (u_n) is nonnegative if and only if, for all n,

   $$B \geq -A(\lambda_1/\lambda_2)^n - Cn - 2|D|\cos(\varphi + n\theta). \tag{13}$$

   If $C < 0$, then (13) cannot be fulfilled. If $C > 0$, then $-A(\lambda_1/\lambda_2)^n - Cn - 2|D|\cos(\varphi + n\theta) \to -\infty$ $(n \to \infty)$. Thus, there exists an explicitly computable $T_2 \in \mathbb{N}$ depending on $A, C, D, \varphi, \theta, \lambda_1, \lambda_2$ such that

   $$\max_{n \in \mathbb{N} \cup \{0\}} \{-A(\lambda_1/\lambda_2)^n - Cn - 2|D|\cos(\varphi + n\theta)\} = -A(\lambda_1/\lambda_2)^{T_2} - CT_2 - 2|D|\cos(\varphi + T_2\theta).$$

   Consequently, the sequence (u_n) is nonnegative if and only if

   $$B \geq -A(\lambda_1/\lambda_2)^{T_2} - CT_2 - 2|D|\cos(\varphi + T_2\theta).$$

III.2.3: *All three roots* λ_1, λ_2 *and* λ_3 *have different absolute values.*

By Proposition 2.1, we need only treat the case where there is a positive dominating root. We consider two possibilities corresponding to the positive dominating root being λ_1 or λ_2.

1. *λ_1 is the positive dominating root, so that $\lambda_1 > |\lambda_2|$, $\lambda_1 > |\lambda_3|$.*

   Here, $u_n = \lambda_1^n \{A + (B + Cn)(\lambda_2/\lambda_1)^n + 2|D|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\}$. The sequence (u_n) is nonnegative if and only if

$$A + (B + Cn)(\lambda_2/\lambda_1)^n + 2|D|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \geq 0 \quad (n \geq 0). \tag{14}$$

   Note that

$$A + (B + Cn)(\lambda_2/\lambda_1)^n + 2|D|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \to A \quad (n \to \infty). \tag{15}$$

   If $A < 0$, then (15) shows that (14) cannot be fulfilled. If $A > 0$, then (15) implies that there is an explicitly computable least integer $T_3 \in \mathbb{N}_0$, depending on $A, B, C, D, \varphi, \theta, \lambda_1, \lambda_2, \lambda_3$ such that (14) holds for all $n \geq T_3$. Consequently, in this case the sequence (u_n) is nonnegative if and only if $T_3 = 0$.

2. *λ_2 is the positive dominating root, so that $\lambda_2 > |\lambda_1|$, $\lambda_2 > |\lambda_3|$.*

   Here, $u_n = \lambda_2^n \{A(\lambda_1/\lambda_2)^n + B + Cn + 2|D|(|\lambda_3|/\lambda_2)^n \cos(\varphi + n\theta)\}$. The sequence (u_n) is nonnegative if and only if

$$B \geq -A(\lambda_1/\lambda_2)^n - Cn - 2|D|(|\lambda_3|/\lambda_2)^n \cos(\varphi + n\theta) \quad (n \geq 0). \tag{16}$$

   - If $C < 0$, then (16) cannot be fulfilled.
   - If $C > 0$, since

$$-A(\lambda_1/\lambda_2)^n - Cn - 2|D|(|\lambda_3|/\lambda_2)^n \cos(\varphi + n\theta) \to -\infty \quad (n \to \infty).$$

     there is an explicitly computable $T_4 \in \mathbb{N}$ depending on $A, C, D, \varphi, \theta, \lambda_1, \lambda_2, \lambda_3$ such that

$$\max_{n \in \mathbb{N} \cup \{0\}} \{-A(\lambda_1/\lambda_2)^n - Cn - 2|D|(|\lambda_3|/\lambda_2)^n \cos(\varphi + n\theta)\} =$$
$$-A(\lambda_1/\lambda_2)^{T_4} - CT_4 - 2|D|(|\lambda_3|/\lambda_2)^{T_4} \cos(\varphi + T_4\theta).$$

     Consequently, the sequence (u_n) is nonnegative if and only if

$$B \geq -A(\lambda_1/\lambda_2)^{T_4} - CT_4 - 2|D|(|\lambda_3|/\lambda_2)^{T_4} \cos(\varphi + T_4\theta).$$

3.2.4 Case IV. C($r_1 r_2 r_3 z \bar{z}$)

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n + \bar{D}\bar{\lambda}_4^n \quad (n \geq 0),$$

where $A, B, C \in \mathbb{R}; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$; $D \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_4 = |\lambda_4|e^{i\theta}$ and $D = |D|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi]$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + 2|D||\lambda_4|^n \cos(\varphi + n\theta).$$

Consider the three real numbers, $\lambda_1, \lambda_2, \lambda_3$. There are two subcases.

IV.1 There are exactly two among $\lambda_1, \lambda_2, \lambda_3$ having the same absolute value, say, $\lambda_2 = -\lambda_3$.

IV.2 None of the $\lambda_1, \lambda_2, \lambda_3$ has the same absolute value.

Subcase IV.1: *There are exactly two among $\lambda_1, \lambda_2, \lambda_3$ having the same absolute value, say, $\lambda_2 = -\lambda_3$.*

Here, $u_n = A\lambda_1^n + \{B + (-1)^n C\}\lambda_2^n + D\lambda_4^n + \bar{D}\bar{\lambda}_4^n$, i.e.,

$$u_{2k} = A\lambda_1^{2k} + \{B + C\}\lambda_2^{2k} + D\lambda_4^{2k} + \bar{D}\bar{\lambda}_4^{2k}$$
$$u_{2k+1} = A\lambda_1^{2k+1} + \{B - C\}\lambda_2^{2k+1} + D\lambda_4^{2k+1} + \bar{D}\bar{\lambda}_4^{2k+1} \quad (k \geq 0).$$

The two sequences (u_{2k}) and u_{2k+1} are decidable because they are of the form (16).

Subcase IV.2: *None of the $\lambda_1, \lambda_2, \lambda_3$ has the same absolute value.*

By Proposition 2.1, we need only treat the case where there is a positive dominating root, say, $\lambda_1 > 0$ and so we may assume without loss of generality that $\lambda_1 > |\lambda_2| > |\lambda_3| > 0$ and $\lambda_1 \geq |\lambda_4|$. We subdivide into two situations depending on whether $\lambda_1 = |\lambda_4|$.

IV.2.1: $\lambda_1 = |\lambda_4|$.

Here, $u_n = \lambda_1^n\{A + (B + C(\lambda_3/\lambda_2)^n)(\lambda_2/\lambda_1)^n + 2|D|\cos(\varphi + n\theta)\} \quad (n \geq 0)$. The sequence (u_n) is nonnegative if and only if

$$A \geq -\{B + C(\lambda_3/\lambda_2)^n\}(\lambda_2/\lambda_1)^n - 2|D|\cos(\varphi + n\theta) \quad (n \geq 0). \tag{17}$$

- If θ is a rational multiple of π, say $\theta = s\pi/t$ where $s, t \in \mathbb{Z}^+ \setminus \{0\}$ and $\gcd(s,t) = 1$. By [11, Lemma 2.2 I], $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ distinct explicit positive and negative values at $n \in \{0, 1, \ldots, 2t-1\}(\text{mod } 2t)$. Since

  $$-\{B + C(\lambda_3/\lambda_2)^n\}(\lambda_2/\lambda_1)^n \to 0 \quad (n \to \infty),$$

  (17) holds if and only if

  $$A \geq \max_{0 \leq n \leq T}\left\{-\{B + C(\lambda_3/\lambda_2)^n\}(\lambda_2/\lambda_1)^n - 2|D|\cos(\varphi + n\theta)\right\}.$$

  for some computable large $T \geq 2t - 1$.

- If θ is not a rational multiple of π, rewrite the term of the sequence as

  $$u_n = |\lambda_2|^n\left\{B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + (A + 2|D|\cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n\right\}.$$

  The sequence $\{u_n\}$ is nonnegative if and only if

  $$B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + (A + 2|D|\cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \geq 0 \qquad (n \geq 0). \tag{18}$$

  We consider four separate situations.

  (a) If $A < 0$, since the values of $\cos(\varphi + n\theta)$ is dense in the closed interval $[-1, 1]$, $(\lambda_2/|\lambda_2|)^n \in \{\pm 1\}$, $(\lambda_3/|\lambda_2|)^n \to 0$, $(\lambda_1/|\lambda_2|)^n \to \infty$ $(n \to \infty)$, there always exists a computable nonnegative integer N^* such that

  $$B(\lambda_2/|\lambda_2|)^{N^*} + C(\lambda_3/|\lambda_2|)^{N^*} + (A + 2|D|\cos(\varphi + N^*\theta))(\lambda_1/|\lambda_2|)^{N^*} < 0$$

  and so (18) cannot be fulfilled.

(b) If $0 < A < 2|D|$, let $\Delta = 2|D| - A > 0$ so that (18) becomes

$$\begin{aligned} & B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + (A + 2|D|\cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \\ & \quad = B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + \{2|D|(1+\cos(\varphi + n\theta)) - \Delta\}(\lambda_1/|\lambda_2|)^n. \end{aligned} \tag{19}$$

Taking a subsequence $\{n_k\}$ for which $\cos(\varphi + n_k\theta) \to -1$ $(k \to \infty)$, the expression in (19) tends to $-\infty$, showing that (18) cannot be fulfilled.

(c) If $A > 2|D| > 0$, let $\delta = A - 2|D| > 0$. Since

$$\{A + 2|D|\cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n \geq \delta(\lambda_1/|\lambda_2|)^n \to \infty \quad (n \to \infty),$$

there is a computable nonnegative integer N^* such that

$$B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + (A + 2|D|\cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \geq 0$$

for all $n \geq N^*$. Thus, (18) holds if and only if $N^* = 0$.

(d) If $A = 2|D|$, then (18) becomes

$$B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + 2|D|(1 + \cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{20}$$

Returning to (18), if there is $N_0 \in \mathbb{N}_0$ such that $1 + \cos(\varphi + N_0\theta) = 0$, which must be unique by [11, Claim 1, p. 140], then for (20) to hold we must have $B(\lambda_2/|\lambda_2|)^{N_0} + C(\lambda_3/|\lambda_2|)^{N_0} \geq 0$. Moreover, using arguments similar to Claim A and the fact that for n large enough the values of $1 + \cos(\varphi + n\theta)$ are positive and bounded by 2, we deduce that

$$B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + 2|D|(1 + \cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N}_0$, such that

$$B(\lambda_2/|\lambda_2|)^n + C(\lambda_3/|\lambda_2|)^n + 2|D|(1 + \cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n \geq 0 \text{ for all } n \geq N_1.$$

Using all the obtained information, we conclude that (20) holds if and only if $N_1 = 0$.

IV.2.2: $\lambda_1 \neq |\lambda_4|$, *and so* $\lambda_1 > |\lambda_4|$.

Here, $u_n = \lambda_1^n\{A + B(\lambda_2/\lambda_1)^n + C(\lambda_3/\lambda_1)^n + 2|D|\cos(\varphi + n\theta)(|\lambda_4|/\lambda_1)^n\}$. The sequence (u_n) is nonnegative if and only if,

$$A + B(\lambda_2/\lambda_1)^n + C(\lambda_3/\lambda_1)^n + 2|D|\cos(\varphi + n\theta)(|\lambda_4|/\lambda_1)^n \geq 0 \quad (n \geq 0). \tag{21}$$

Next note that

$$A + B(\lambda_2/\lambda_1)^n + C(\lambda_3/\lambda_1)^n + 2|D|\cos(\varphi + n\theta)(|\lambda_4|/\lambda_1)^n \to A \quad (n \to \infty). \tag{22}$$

If $A < 0$, then (22) shows that (21) cannot be fulfilled. If $A > 0$, then (22) implies that there is an explicitly computable least integer $N_2 \in \mathbb{N}_0$, depending on $A, B, C, D, \varphi, \theta, \lambda_1, \lambda_2, \lambda_3$ such that (21) holds for all $n \geq N_2$. Consequently, in this case the sequence (u_n) is nonnegative if and only if $N_2 = 0$.

3.2.5 Case V. C($rz_1\bar{z}_1z_2\bar{z}_2$)

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0),$$

where $A, \lambda_1 \in \mathbb{R}\setminus\{0\}$; $B, C \in \mathbb{C}$ and $\lambda_2, \lambda_3 \in \mathbb{C}\setminus\mathbb{R}$. Let $\lambda_2 = |\lambda_2|e^{i\theta_1},\ \lambda_3 = |\lambda_3|e^{i\theta_2},\ B = |B|e^{i\varphi_1}$ and $C = |C|e^{i\varphi_2}$ where $\theta_1, \theta_2, \varphi_1, \varphi_2 \in [-\pi, \pi]$, $\theta_1, \theta_2 \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + 2|B||\lambda_2|^n \cos(\varphi_1 + n\theta_1) + 2|C||\lambda_3|^n \cos(\varphi_2 + n\theta_2).$$

By Proposition 2.1, we need only treat the case where there is a positive dominating root, i.e., $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|\} > 0$. We assume that all the three coefficients A, B and C are nonzero for otherwise we are left with the case (15) or the case (7) in [11, Lemma 2.3]. There are three subcases.

V.1 The three λ's have the same absolute values, i.e., $\lambda_1 = |\lambda_2| = |\lambda_3|$.

V.2 All three roots $\lambda_1,\ \lambda_2$ and λ_3 have different absolute values.

V.3 There are exactly two λ_i's having the same absolute value, i.e., $\lambda_1 = |\lambda_2|$ or $\lambda_1 = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.

Subcase V.1: $\lambda_1 = |\lambda_2| = |\lambda_3|$.
Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + 2|B|\cos(\varphi_1 + n\theta_1) + 2|C|\cos(\varphi_2 + n\theta_2)\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if

$$A + 2|B|\cos(\varphi_1 + n\theta_1) + 2|C|\cos(\varphi_2 + n\theta_2) \geq 0 \quad (n \geq 0). \tag{23}$$

Clearly, the sign of this last expression depends on the nature of θ_1 and θ_2. We have three possibilities.

- Both θ_1 and θ_2 are rational multiples of π, say $\theta_1 = s_1\pi/t_1$ and $\theta_2 = s_2\pi/t_2$ where $s_1, s_2 \in \mathbb{Z}\setminus\{0\}$, $t_1, t_2 \in \mathbb{N}$, $\gcd(s_1, t_1) = 1$ and $\gcd(s_2, t_2) = 1$. By [11, Lemma 2.2 I], $\cos(\varphi_1 + n\theta_1)$ is periodic, taking at most $2t_1$ explicitly computable positive and negative values corresponding to $n = 0, 1, \ldots, 2t_1 - 1$, and $\cos(\varphi_2 + n\theta_2)$ is periodic, taking at most $2t_2$ explicitly computable positive and negative values corresponding to $n = 0, 1, \ldots, 2t_2 - 1$. Then, (23) holds if and only if

$$A \geq \max_{0 \leq n \leq \operatorname{lcm}(2t_1, 2t_2)} \{-2|B|\cos(\varphi_1 + n\theta_1) - 2|C|\cos(\varphi_2 + n\theta_2)\theta\}$$

- One of the angles, say, θ_1 is a rational multiple of π, while the other, θ_2 is not a rational multiple of π. Let $\theta_1 = s\pi/t$ where $s, t \in \mathbb{Z}^+\setminus\{0\}$, $\gcd(s, t) = 1$, By [9, Lemma 2.2 I, III], the function $\cos(\varphi_1 + n\theta_1)$ is periodic and takes at most $2t$ explicitly computable positive and negative values corresponding to $n = 0, 1, \ldots, 2t - 1$. Let the minimal value $\mathcal{B} := \min_{n\in\mathbb{N}_0}\{2|B|\cos(\varphi_1 + n)\theta_1\}$ occurs at $T \pmod{2t}$. Taking $n = T + 2kt$ $(k \geq 0)$, the result [11, Lemma 2.2 II] tells us that the values of $\cos(\varphi_2 + (T + 2kt)\theta_2)$ is dense in $[-1, 1]$. Consequently, the condition (23) holds for all $n \geq 0$ if and only if $A + \mathcal{B} - 2|C| \geq 0$.

- Both $\theta_1 := 2\pi\tilde{\theta}_1$ and $\theta_2 := 2\pi\tilde{\theta}_2$ are not rational multiples of π.

  (a) If $1, \tilde{\theta}_1, \tilde{\theta}_2$ are linearly independent over $\mathbb{Q}$, then by the Kroneker-Weyl theorem, [4, Theorems IV.I and IV.II], the value set of all the pairs of fractional parts $((n\tilde{\theta}_1), (n\tilde{\theta}_2))$ is dense in the unit square $[0, 1) \times [0, 1)$. Thus, the condition (23) holds for all $n \geq 0$ if and only if $A - 2|B| - 2|C| \geq 0$.

  (b) If $1, \tilde{\theta}_1, \tilde{\theta}_2$ are linearly dependent over $\mathbb{Q}$, then there exist $m_0, m_1, m_2 \in \mathbb{Z} \setminus \{0\}$ such that $\tilde{\theta}_2 = \dfrac{m_0 + m_1\tilde{\theta}_1}{m_2}$. Let $n = m_2 N + J$, where $N \in \mathbb{N}_0,\ 0 \leq J \leq m_2 - 1$. Consider,

  $$C_J(N) = A + 2|B|\cos(\tilde{\varphi}_1(J) + 2m_2 N\pi\tilde{\theta}_1) + 2|C|\cos(\tilde{\varphi}_2(J) + 2m_1 N\pi\tilde{\theta}_1)$$

  where $\tilde{\varphi}_1(J) = \varphi_1 + 2J\pi\tilde{\theta}_1$ and $\tilde{\varphi}_2(J) = \varphi_2 + 2J\pi(m_0 + m_1\tilde{\theta}_1)/m_2$. Let

  $$f_J(t) = A + 2|B|\cos(\tilde{\varphi}_1(J) + 2m_2\pi t) + 2|C|\cos(\tilde{\varphi}_2(J) + 2m_1\pi t),$$

  where $t \in [0, 1]$. Since $f_J(t)$ is continuous over the compact set $[0, 1]$, the minimum $\min_{t\in[0,1]} f_J(t) = M_J(f)$ exists, say, at t_0. Since the set of fractional parts $\left\{(N\tilde{\theta}_1)\right\}$ is dense in $[0, 1]$, there exists subsequence $(N_k) \subseteq \mathbb{N}$ such that $(N_k\tilde{\theta}_1) \to t_0$ and so $C_J(N_k) \downarrow M_J(f)$, which implies that $C_J(N) \geq 0$ if and only if $M_J(f) \geq 0$. Therefore, the sequence (u_n) is nonnegative if and only if $\min_{J\in\{0,1,\ldots,m_2-1\}} M_J(f) \geq 0$.

Subcase V.2: *All three roots λ_1, λ_2 and λ_3 have different absolute values.*

Without loss of generality assume $\lambda_1 > |\lambda_2| > |\lambda_3|$. Here,

$$u_n = \lambda_1^n\{A + 2\left(|B|\cos(\varphi_1 + n\theta_1) + |C|(|\lambda_3|/|\lambda_2|)^n\cos(\varphi_2 + n\theta_2)\right)(|\lambda_2|/\lambda_1)^n\} \quad (n \geq 0),$$

The sequence (u_n) is nonnegative if and only if

$$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|(|\lambda_3|/|\lambda_2|)^n\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \geq 0 \qquad (n \geq 0).$$

- If $A < 0$, then $A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|(|\lambda_3|/|\lambda_2|)^n\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n < 0$ for all sufficiently large n, and so this case is untenable.

- If $A > 0$, since

  $$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|(|\lambda_3|/|\lambda_2|)^n\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \to A > 0\ (n \to \infty),$$

  there is an explicitly computable least $M_1 \in \mathbb{N}_0$, depending on $A, B, C, \lambda_1, \lambda_2, \lambda_3\ \theta_1, \theta_2, \varphi_1, \varphi_2$ for which

  $$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|(|\lambda_3|/|\lambda_2|)^n\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \geq 0 \ \text{ for all } n \geq M_1.$$

  Thus, the sequence (u_n) is nonnegative if and only if $M_1 = 0$.

Subcase V.3: $\lambda_1 = |\lambda_2|$ *or* $\lambda_1 = |\lambda_3|$ *or* $|\lambda_2| = |\lambda_3|$.

We need only treat two possibilities.

V.3.1: $\lambda_1 = |\lambda_2|$ *or* $\lambda_1 = |\lambda_3|$.

Without loss of generality, let $\lambda_1 = |\lambda_2| > |\lambda_3|$. Here,

$$u_n = \lambda_1^n \left\{A + 2|B|\cos(\varphi_1 + n\theta_1) + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi_2 + n\theta_2)\right\}.$$

Then, the sequence (u_n) is nonnegative if and only if

$$A \geq -2|B|\cos(\varphi_1 + n\theta_1) - 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi_2 + n\theta_2) \qquad (n \geq 0). \tag{24}$$

- If θ_1 is a rational multiple of π, say $\theta_1 = s\pi/t$ where $s, t \in \mathbb{Z}^+ \setminus \{0\}$ and $\gcd(s,t) = 1$. By [9, Lemma 2.2], $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ distinct explicit (positive and negative) values at $n \in \{0, 1, \ldots, 2t-1\} (\mod 2t)$. Let $P := \{p_1, \ldots, p_s\}$, $Q := \{q_1, \ldots, q_r\}$ be subset of $\{0, 1, \ldots, 2t-1\}$ such that

  $$c_i = \cos(\varphi_1 + n\theta_1) < 0 \ (1 \leq i \leq s), \quad d_j = \cos(\varphi_1 + n\theta_1) \geq 0 \ (1 \leq j \leq r)$$

  Since $\mathbb{N}_0 = (p_1 + 2t\mathbb{N}_0) \cup \cdots \cup (p_s + 2t\mathbb{N}_0) \cup (q_1 + 2t\mathbb{N}_0) \cup \cdots \cup (q_r + 2t\mathbb{N}_0)$, and for each $\ell \in \mathbb{N}_0$, let

  $$F_i(\ell) = -2|B|c_i - 2|C|(|\lambda_3|/\lambda_1)^{p_i+2t\ell} \cos(\varphi_2 + (p_i + 2t\ell)\theta_2) \quad (i = 1, \ldots, s)$$
  $$N_j(\ell) = -2|B|d_j - 2|C|(|\lambda_3|/\lambda_1)^{q_j+2t\ell} \cos(\varphi_2 + (q_j + 2t\ell)\theta_2) \quad (j = 1, \ldots, r)$$

  If $2|C|(|\lambda_3|/\lambda_1)^{p_i+2t\ell} \cos(\varphi_2 + (p_i + 2t\ell)\theta_2) \leq 0$ for all $\ell \in \mathbb{N}_0$, then we see clearly that $\sup_{\ell\in\mathbb{N}_0} F_i(\ell) = -2|B|c_i$. If there exists $\ell_0 \in \mathbb{N}_0$ such that

  $$-2|C|(|\lambda_3|/\lambda_1)^{p_i+2t\ell_0} \cos(\varphi_2 + (p_i + 2t\ell_0)\theta_2) > 0,$$

  since $-2|C|(|\lambda_3|/\lambda_1)^{p_i+2t\ell_0} \cos(\varphi_2 + (p_i + 2t\ell_0)\theta_2) \to 0$ $(\ell_0 \to \infty)$, there is an $L_i \in \mathbb{N}_0$, $L_i \geq \ell_0$ for which

  $$\sup_{\ell\in\mathbb{N}_0} F_i(\ell) = -2|B|c_i + \max_{\ell\in\{0,1,\ldots,L_i\}} \left\{-2|C|(|\lambda_3|/\lambda_1)^{p_i+2t\ell} \cos(\varphi_2 + (p_i + 2t\ell)\theta_2)\right\};$$

  call this maximum M_i. Similarly, if $-2|C|(|\lambda_3|/\lambda_1)^{q_j+2t\ell} \cos(\varphi_2 + (q_j + 2t\ell)\theta_2) \leq 0$ for all $\ell \in \mathbb{N}_0$, then $\sup_{\ell\in\mathbb{N}_0} N_j(\ell) = -2|B|d_j$, while if there exists $\ell_1 \in \mathbb{N}_0$ such that

  $$-2|C|(|\lambda_3|/\lambda_1)^{q_j+2t\ell} \cos(\varphi_2 + (q_j + 2t\ell)\theta_2) > 0,$$

  then $\max_{\ell\in\mathbb{N}_0} N_j(\ell)$ is attainable; call this maximum K_j. Thus, (24) holds if and only if

  $$A \geq \max_{1\leq i\leq s, 1\leq j\leq r} \{-2|B|c_i, M_i, K_j\}$$

  and so this case is decidable.

- If θ_1 is not a rational multiple of π, rewrite the terms of the sequence as

  $$u_n = |\lambda_3|^n \left\{(A + 2|B|\cos(\varphi_1 + n\theta_1))(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi_2 + n\theta_2)\right\}.$$

  The sequence $\{u_n\}$ is nonnegative if and only if

  $$(A + 2|B|\cos(\varphi_1 + n\theta_1))(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi_2 + n\theta_2) \geq 0 \quad (n \geq 0) \tag{25}$$

  We consider four separate situations.

(a) If $A < 0$, since the values of $\cos(\varphi_1 + n\theta_1)$ is dense in the closed interval $[-1, 1]$, and $(\lambda_1/|\lambda_3|)^n \to \infty$ $(n \to \infty)$, there is a computable nonnegative integer N_A such that

$$(A + 2|B|\cos(\varphi_1 + N_A\theta_1))\,(\lambda_1/\,|\lambda_3|)^{N_A} + 2\,|C|\cos(\varphi_2 + N_A\theta_2) < 0$$

and so (25) cannot be fulfilled.

(b) If $0 < A < 2\,|B|$, let $\Delta = 2\,|B| - A > 0$ so that

$$\begin{aligned}&(A + 2|B|\cos(\varphi_1 + n\theta_1))\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2)\\&= \{2\,|B|\,(1 + \cos(\varphi_1 + n\theta_1) - \Delta)\}\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2).\end{aligned} \tag{26}$$

Taking a subsequence $\{n_k\}$ for which $\cos(\varphi_1 + n_k\theta_1) \to -1$ $(k \to \infty)$, the left-hand side of (26) tends to $-\infty$ as $k \to \infty$, showing that (25) cannot be fulfilled.

(c) If $A > 2\,|B| > 0$, let $\delta = A - 2\,|B| > 0$. Since

$$\begin{aligned}&(A + 2|B|\cos(\varphi_1 + n\theta_1))(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2)\\&\qquad\geq \delta(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2) \to \infty \quad (n \to \infty),\end{aligned}$$

there is a computable nonnegative integer N^* depending on $A, B, C, \varphi_1, \varphi_2, \theta_1, \theta_2$ such that

$$(A + 2|B|\cos(\varphi_1 + n\theta_1))\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2) \geq 0$$

for all $n \geq N^*$. Thus, (25) holds if and only if $N^* = 0$.

(d) If $A = 2\,|B|$, then (25) becomes

$$2\,|B|\,(1 + \cos(\varphi_1 + n\theta_1))\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2) \geq 0 \quad (n \geq 0). \tag{27}$$

If there is $N_0 \in \mathbb{N}_0$ such that $1 + \cos(\varphi_1 + N_0\theta_1) = 0$, which must be unique by [11, Claim 1, p. 140], then for (27) to hold we must have $2\,|C|\cos(\varphi_2 + N_0\theta_2) \geq 0$. Moreover, using arguments similar to [8, Lemma 2.2], we deduce that

$$2\,|B|\,\{1 + \cos(\varphi_1 + n\theta_1)\}\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2) \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N}_0$, depending on $B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$2\,|B|\,\{1 + \cos(\varphi_1 + n\theta_1)\}\,(\lambda_1/\,|\lambda_3|)^n + 2\,|C|\cos(\varphi_2 + n\theta_2) \geq 0 \;\text{ for all } n \geq N_1.$$

Using all the obtained information, we conclude that (25) holds if and only if $N_1 = 0$.

V.3.2: $|\lambda_2| = |\lambda_3|$ $(< \lambda_1)$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\,\{A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n\}\,.$$

The sequence (u_n) is nonnegative if and only if

$$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \geq 0 \quad (n \geq 0). \tag{28}$$

- If $A < 0$, then $A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n < 0$ for all sufficiently large n, and so (28) is untenable.
- If $A > 0$, since

$$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \to A > 0 \ (n \to \infty),$$

there is an explicitly computable least $M_2 \in \mathbb{N}_0$, depending on $A, B, C, \lambda_1, \lambda_2, \theta_1, \theta_2, \varphi_1, \varphi_2$ for which

$$A + 2\{|B|\cos(\varphi_1 + n\theta_1) + |C|\cos(\varphi_2 + n\theta_2)\}(|\lambda_2|/\lambda_1)^n \geq 0 \quad \text{for all } n \geq M_2.$$

Thus, the sequence (u_n) is nonnegative if and only if $M_2 = 0$.

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