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# USE OF PADE APPROXIMATION FOR $e^{-z}$ TO APPROXIMATE AN INTEGRAL OF A BESSEL FUNCTION

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*


## SUMMARY

In a number of my previous works, I have emphasized a global approach to the approximation of functions. Numerical values of a function are but a facet of the overall problem. We desire approximations to evaluate functions and their zeros, to simplify transforms and their inverses, and to facilitate the direct solution of various functional equations, In this paper, we illustrate our ideas by using Padé approximations for the exponential function to evaluate an integral involving the Bessel function $I_0(z)$.

## RESUMEN

En mis numerosos trabajos anteriores, se ha enfatizado sobre un enfoque global a la aproximación de las funciones. Los valores numéricos de una función son solamente una parte del problema. Desear obtener aproximaciones para evaluar las funciones, sus ceros, simplificar las transformadas (y sus inversas), y facilitar la solución directa de varias ecuaciones funcionales. En este trabajo, se ilustran nuestras ideas usando aproximaciones Padé para la función exponencial para evaluar una integral involucionando la función de Bessel $I_0$ (z).


## INTRODUCTION

In this paper we illustrate use of Padé approximations for $e^{-z}$ to achieve rather simple approximations for

$$J(x,y) = 1 - e^{-y} \int_0^x e^{-t} I_0\{2(yt)^{1/2}\} dt \qquad ...(1)$$

where $I_0(z)$ is the modified Bessel function of order zero. This integral arises in many physical and mathematical applications. We do not give a detailed bibliography, but see Luke [1], Bolshev and Kuznetsov [2] and Lassey [3] and the references given in these sources. See also Lightfoot [4] and Watson [5]. We follow the notation in [1]. It is sufficient to consider the situation $0 < x \leq y$ because

$$J(x,y) + J(y,x) = 1 + e^{-(x+y)} I_0(\xi), \quad \xi = 2(xy)^{1/2} \ldots (2)$$

and $I_0(\xi)$ is easy to evaluate in view of the approximations and expansions given in my works [6], [7] and [8].

If G(x,p) is the Laplace transform of J (x,y), that is,

$$G(x,p) = \int_0^\infty e^{-py} J(x,y) dy \qquad ...(3)$$

it is easy to show that

$$G(x,p) = \frac{1}{p} \exp\left(-\frac{px}{p+1}\right) \qquad ...(4)$$

Our idea is to use Padé approximations for $e^{-z}$ to approximate G(x,p). We only conseder those Padé approximations wich lie on or below the main diagonal so that the approximations for G(x,p) can be expressed as the ratio of two polynomials in p where the degree of the denominator polynomial exceeds that of the numerator polynomial by at least one. By partial fration decomposition and ordinary inversion of Laplace transforms, we get approximations for J(x,y) in the form of sums of products of exponential functions and trigonometric functions.

## APPROXIMATIONS FOR G(x,p) AND J(x,y)

For Padé approximations to the exponential function we write

$$e^{-z} = \frac{A_m(z)}{B_n(z)} + R_{n,m}(z) \qquad ...(5)$$

where m and n are the respective degrees of the polynomials $A_m(z)$ and $B_n(z)$, $n \geq m$. Thus $A_m(z)/B_n(z)$ is the Padé approximation to $e^{-z}$ and $R_{m,n}(z)$ is the remainder. If m = n or m = n-1, many details on these approximations are given in [6], [7] and [8]. Corresponding data for the entire spectrum of m and n are presented in Luke [9]. Our presentation is quite general, though for the numbers we consider only the cases m = n and m = n-1. We can write

$$\frac{1}{z}\frac{A_m(z)}{B_n(z)} = \sum_{k=0}^{n} \frac{C_k}{z+\alpha_k}, \alpha_0 = 0, C_0 = 1,$$

$$C_k = -\frac{A_m(-\alpha_k)}{\alpha_k B_n'(-\alpha_k)}, \; k = 1, 2, \ldots, n, \qquad \ldots(6)$$

where the $\alpha_k$'s, $k = 1, 2, \ldots, n$ are the zeros of $B_n(-z)$. If $m = n$, extensive tables of $\alpha_k$ and $C_k$ have been given by Krylov and Skoblja [10]. When $m = n$ or $m = n-1$, the zeros are complex conjugates except when n is odd in which case one zero is real. If n is odd, we let $\alpha_n$ be the real zero of $B_n(-z)$.

Let

$$G_{m,n}(x,p) = \frac{1}{p}\frac{A_m(z)}{B_n(z)}, \; z = \frac{px}{p+1} \qquad \ldots(7)$$

Then, we have

$$G_{m,n}(x,p) = 1 + \sum_{k=1}^{n} \frac{[x/(x+\alpha_k)]C_k}{\left[p + \frac{\alpha_k}{x+\alpha_k}\right]} \qquad \ldots(8)$$

Put

$\alpha_k = \beta_k + i\gamma_k$, $k = 1,3,\ldots,2r-1$

$\alpha_k = \beta_k - i\gamma_k$, $k = 2,4,\ldots,2r$ if $n = 2r$ is even; ..(9)

and

$\alpha_k = \beta_k + i\gamma_k$, $k = 1,3,\ldots,2r-3$

$\alpha_k = \beta_k - i\gamma_k$, $k = 2,4,\ldots,2r-2$ if $n = 2r-1$ is odd. ...(10)

Recall that if n is odd, $\alpha_n$ is real.

Further, let

$$g_k(x) = (x+\beta_k)^2 + \gamma_k^2, \qquad \ldots(11)$$

$$\delta_k = \frac{\beta_k(x+\beta_k)+\gamma_k^2}{g_k(x)}, \qquad \theta_k = \frac{\gamma_k x}{g_k(x)} \qquad \ldots(12)$$

$$\left(\frac{x}{x+\alpha_k}\right) C_k = \left(\frac{x}{x+\alpha_k}\right)(D_k + iE_k) = U_k + iV_k, \qquad \ldots(13)$$

$$U_k = \frac{x[(x+\beta_k)D_k + \gamma_k E_k]}{g_k(x)}, \quad V_k = \frac{x[(x+\beta_k)E_k - \gamma_k D_k]}{g_k(x)} \qquad (14)$$

Then if $J_{m,n}(x,y)$ is the inverse Laplace transform of $G_{m,n}(x,p)$, $J_{m,n}(x,y)$ is an approximation to $J(x,y)$ and we have

$$J_{m,n}(x,y) = 1 + 2\sum_{s=1}^{r} e^{-\delta_{2s-1}y}\left(U_{2s-1}\cos\theta_{2s-1}y + V_{2s-1}\sin\theta_{2s-1}y\right), \; n = 2r. \qquad \ldots(15)$$

If n is odd, $n = 2r-1$, the latter also holds with $\alpha_n = \beta_n$, $\gamma_n = 0$ (whence $\theta_n = 0$), $E_n = 0$, and with the proviso that $U_n$ is taken with half weight.

## III. ERROR ANALYSIS

In this section, we discuss the error in the approximations for $J(x,y)$. From the theory of inverse Laplace transforms, we have

$$J(x,y) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} p^{-1}e^{py}e^{-px/(p+1)}dp, \qquad \ldots(16)$$

$$J_{m,n}(x,y) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} p^{-1}e^{py}\{A_m(z)/B_n(z)\}dp. \qquad (17)$$

Thus if $S_{m,n}(x,y)$ is the error in the approximations for $J(x,y)$,

$$S_{m,n}(x,y) = J(x,y) - J_{m,n}(x,y), \qquad \ldots(18)$$

then

$$S_{m,n}(x,y) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} p^{-1}e^{py}R_{m,n}(z)dp \qquad \ldots(19)$$

In (16)-(18), $c \geq 0$ and without loss of generality, we can take $c = 0$. Forms for $R_{m,n}(z)$ when $m = n$ or $m = n-1$ are given in [6] and [7]. When $m = n$, this information is also available in [8]. For the general case, see [9]. Now from these sources, omitting terms of $O(n^{-1})$, we have for $m = n$ or $m = n-1$.

$$R_{m,n}(z) = k_{m,n}z^{m+n+1}e^{-z},$$

$$k_{m,n} = (-)^{m+1}\left[2^{2n+2m+1}n!m!\right]^{-1} \qquad \ldots(20)$$

In (19), let $p = iq$. Then we can write

$$S_{m,n}(x,y) = (-)^{m+1}x^{m+n+1}k_{m,n}\left[\int_0^\infty \frac{1}{q}\left(\frac{q}{q^2+1}\right)^{m+n+1} e^{-q^2x/(q^2+1)}\right] X \left\{e^{iqy}e^{-iqx/(q^2+1)}(q+i)^n - e^{-iqy}e^{iqx/(q^2+1)}(q-i)^n\right\}dq. \qquad \ldots(21)$$

With x fixed, a rather straight forward argument shows that

$$\lim_{n\to\infty} S_{m,n}(x,y) = 0 \qquad \ldots(22)$$

Further, the convergence is quite rapid in view of the structure of $k_{m,n}$. We omit details primarily because the analysis does not seem to lead to useful a priori estimates of the error. The statement (22) can be shown to hold for all $m \leq n$. We can achieve an a posteriori analysis of the error as follows. As previously noted, we can take $0 < x \leq y$. Now if n is fixed, the relative error in the Padé approximation for $e^{-z}$ increases montonically as z increases. In the integrand of (19), p goes from 0 to $i\infty$ whence z goes from 0 to x. So the magnitude of the integrand of (19) is largest when z=x,n sufficiently large. Since the integrand is oscillatory, $S_{m,n}(x,y)$ does not follow this type of behavior. Indeed, as might be expected, the numerics confirm that for y and n fixed, n sufficiently large, $S_n(x,y)$ is oscillatory as x increases from 0 to y. We conjecture that the maxima of $S_{m,n}(x,y)$ do not exceed the magnitude of the error in the Padé approximation for $e^{-x}$. Actually, this estimate is much conservative.

From (2),

$$J(x,x) = \frac{1}{2}\left[1 + e^{-2x} I_0(2x)\right] \qquad \ldots(23)$$

which is readily evaluated. Thus once m and n are selected and $J_m(x,y)$ is deduced, we can easily compute

$$S_{m,n}(x,x) = \frac{1}{2}\left[1 + e^{-2x} I_0(2x)\right] - J_m(x,x) \quad \ldots(24)$$

and the magnitudes of the curve $S_{m,n}(x,x)$ should be a bound or near to that for $S_{m,n}(x,y)$ for all $0<x\leq y$

NUMERICS

In this section we present the approximations $J_{2,2}(x,y)$, $J_{2,3}(x,y)$, $J_{4,4}(x,y)$ and $J_{4,5}(x,y)$ along with numerics for x= y.

**1. m = n = 2.**

$B_2(z) = z^2+6z+12$, $A_2(z)=B_2(-z)$.

$$J_{2,2}(x,y) = 1 - \frac{4x}{g(x)}\left(\exp\left[-\frac{3(x+4)y}{g(x)}\right]\right)$$

$$\left[3\cos\theta y+(x+3)\sqrt{3}\sin\theta y\right],$$

$$\theta = \frac{x\sqrt{3}}{g(x)}, \quad g(x) = x^2 + 6x + 12.$$

| x | $J_{2,2}(x,x)$ | $J(x,x)$ | $S_2(x,x)$ |
|---|---|---|---|
| 0.2 | .84870 119 | .84870 108 | -.11(-6) |
| 0.4 | .76207 380 | .76207 447 | .67(-6) |
| 0.5 | .73287 544 | .73287 980 | .44(-5) |
| 1.0 | .65410 791 | .65425 416 | .15(-3) |
| 2.0 | .60272 205 | .60350 096 | .78(-3) |
| 4.0 | .57533 242 | .57171 589 | -.36(-2) |
| 5.0 | .57255 979 | .56391 667 | -.86(-2) |
| 10.0 | .58250 533 | .54439 016 | -.38(-1) |

2. m= 2, n= 3

$B_3(z)= z^3 + 9z^2 + 36z + 60$, $A_2(z)= 3z^2 - 24z + 60$

$\beta_1= 2.68108\ 2874$, $\gamma_1= 3.05043\ 0199$

$\alpha_3= \beta_3= 3.63783\ 4253$, $\gamma_3= 0$

$g_1(x)= (x+\beta_1)^2+\gamma_1^2= x^2+5.36216\ 5748x+16.49332\ 978$

$$\delta_1=\frac{\beta_1 x + 16.49332\ 978}{g_1(x)}, \quad \theta_1= \frac{\gamma_1 x}{g_1(x)}$$

$$U_1= \frac{x}{g_1(x)}\left[2.01488\ 8929x + 3.15541\ 9317\right],$$

$$V_1= -\frac{x}{g_1(x)}\left[.73650\ 75538x + 8.12091\ 5826\right].$$

$$\delta_3= \frac{\beta_3 x}{x+\beta_3}, \quad \theta_3= 0,$$

$$U_3= -\frac{5.02977\ 7858x}{x+\beta_3}$$

Note: In (15), insert $U_3$ with half weight.

| x | $J_{2,3}(x,x)$ | $S_{2,3}(x,x)$ |
|---|---|---|
| .4 | .76207 441 | .6(-7) |
| 1.0 | .65424 288 | .113(-4) |
| 2.0 | .60346 646 | .345(-4) |
| 5.0 | .56675 615 | -.284(-2) |
| 10.0 | .55727 910 | -.124(-1) |

3. m= n= 4

$B_4(z)= z^4 + 20z^3 + 180z^2 + 840z\cdot +1680$, $A_4(z)=B_4(-z)$

$\beta_1= 4.20757\ 8794$, $\gamma_1= 5.31483\ 6084$

$\beta_3= 5.79242\ 1206$ $\gamma_3= 1.73446\ 8258$

$g_1(x)= (x+\beta_1)^2+\gamma_1^2= x^2+8.41515\ 7588x+45.95120\ 191$

$g_3(x)= (x+\beta_3)^2+\gamma_3^2= x^2+11.58484\ 2414x+36.56052\ 357$

$$\delta_1= \frac{\beta_1 x + 45.95120\ 191}{g_1(x)}, \quad \theta_1= \frac{\gamma_1 x}{g_1(x)}$$

$$\delta_3= \frac{\beta_3 x + 36.56052\ 357}{g_1(x)}, \quad \theta_3= \frac{\gamma_3 x}{g_3(x)}$$

$$U_1 = \frac{x(-2.28774\ 9291x + 27.06817\ 651)}{g_1(x)}$$

$$V_1 = \frac{x(6.90408\ 1581x + 41.20847\ 974)}{g_1(x)}$$

$$U_3 = \frac{x(2.28774\ 9291x - 39.81673\ 224)}{g_3(x)}$$

$$V_3 = -\frac{x(30.59631\ 649x + 181.19478\ 10)}{g_3(x)}$$

| x | $J_{4,4}(x,x)$ | $S_{4,4}(x,x)$ |
|---|---|---|
| 1.0 | .65425 4163 | 0 |
| 2.0 | .60362 7357 | -.401(-6) |
| 4.0 | .57150 0967 | .149(-4) |
| 5.0 | .56385 1269 | .654(-4) |
| 10.0 | .54481 9748 | .704(-4) |
| 15.0 | .53886 1742 | -.229(-2) |

4. $m = 4,\ n = 5.$

$B_5(z) = z^5+25z^4+300z^3+2100z^2+8400z+15120$

$A_4(z) = 5z^4-120z^3+1260z^2-6720z+15120$

$\beta_1 = 3.65569\ 4325,\quad \gamma_1 = 6.54373\ 6899$

$\beta_3 = 5.70095\ 3299,\quad \gamma_3 = 3.21026\ 5600$

$\alpha_5 = \beta_5 = 6.28670\ 4752,\quad \gamma_5 = 0$

$$g_1(x) = (x+\beta_1)^2 + \gamma_1^2 = x^2+7.31138\ 8650x+56.18459\ 360$$

$$g_3(x) = (x+\beta_3)^2 + \gamma_3^2 = x^2+11.40190\ 6598x+42.80667\ 374$$

$$\delta_1 = \frac{\beta_1 x+56.18459\ 360}{g_1(x)},\qquad \theta_1 = \frac{\gamma_1 x}{g_1(x)}$$

$$\delta_3 = \frac{\beta_3 x+42.80667\ 374}{g_3(x)},\qquad \theta_3 = \frac{\gamma_3 x}{g_3(x)}$$

$$\delta_5 = \frac{\beta_5 x}{x+\beta_5},\qquad \theta_5 = 0$$

$$U_1 = -\frac{x(3.83966\ 1632x + 12.24645\ 6541)}{g_1(x)}$$

$$V_1 = \frac{x(.27357\ 03963x + 26.12582\ 525)}{g_1(x)}$$

$$U_3 = \frac{x(25.07945\ 218x + 135.95512\ 54)}{g_3(x)}$$

$$V_3 = -\frac{x(2.18725\ 2125 + 92.98112\ 482)}{g_3(x)}$$

$$U_5 = -\frac{43.47958\ 095x}{x + \beta_5}.$$

Note: In (15), take $U_5$ with half weigh

| | $J_{4,5}(x,x)$ | $S_{4,5}(x,x)$ |
|---|---|---|
| 1 | .65425 418 | -.2(-7) |
| 2 | .60350 102 | -.6(-7) |
| 4 | .57171 279 | .310(-5) |
| 5 | .56390 276 | .139(-4) |
| 10 | .54499 895 | -.109(-4) |
| 15 | .53765 766 | -.109(-2) |

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# INTEGRATION OVER A HYPERCUBE

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*

The major difficulty one usually encounters in evaluating iterated integrals is the determination of the limits of integration.

Evaluate $\displaystyle\int_{0\le x_1x_2+x_3x_4\le 1,} x_1x_2^2x_3^2x_4^2\,dx_1dx_2dx_3dx_4$

$0\le x_k\le 1,\ k=1,2,3,4$

In order to get an idea on how to determine just what portions of the hypercube $0\le x_k\le 1$, k=1,2,3,4 constitute the region of integration, consider an analogous problem in ordinary three-space using $x_1$, $x_2$, $x_3$ for x, y, z. Then we will do the four-space problem by analogy. So, as a preliminary problem evaluate

$$\iiint_T x_1x_2^2x_3^2\,dV$$

where T denotes the region defined by

$$T:\begin{cases}0\le x_1x_2+x_1x_3\le 1\\ 0\le x_k\le 1,\ k=1,2,3\end{cases}$$

The region T is some portion (or portions) of the cube C: $0\le x_k\le 1$, k= 1,2,3. To find out what portion or portions of C make up the region T, let us first put $x_1$ at its maximum possible value of 1 Thus the intersection of the upper plane of the cube C by the surface S: $x_1x_2+x_1x_3=1$ gives the line L: $x_1=1$, $x_2+x_3=1$ as in Fig. I.

The line L is obtained by taking $x_1$ at its maximum possible value. We see, therefore, that the entire half of the cube lying on the same side of the plane $x_2+x_3=1$ as the origin makes up part of the region of integration T, namely, all those points in space for which $0\le x_1\le 1$, $0\le x_2+x_3\le 1$.

There remains the question: What part of the other half of cube C belongs to T? To obtain the answer to this question one must observe that in this half of cube C we must have $x_2+x_3>1$. This means that $x_1$ must now be sufficiently less than unity as will satisfy the requirement $x_1x_2+x_1x_3\le 1$, wich in turn means that we can admit only those points of the cube C that lie on or below the surface S: $x_1x_2+x_1x_3=1$ as shown in Fig. II.

Thus, the region of integration T is made up of the prism resting on the triangle POQ in Fig. I

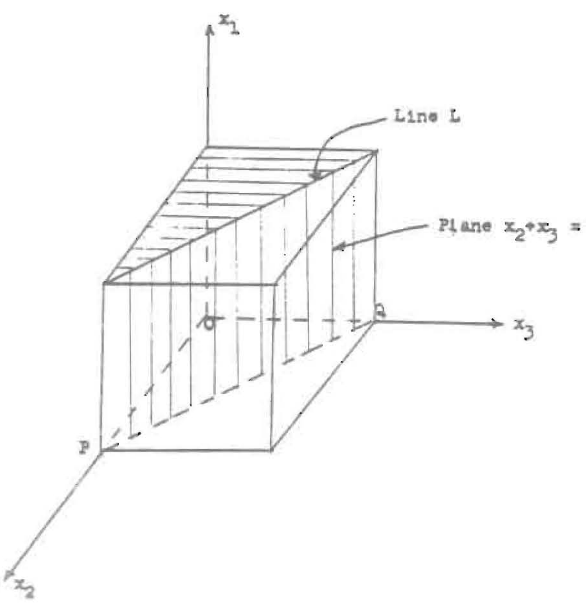


Fig. I

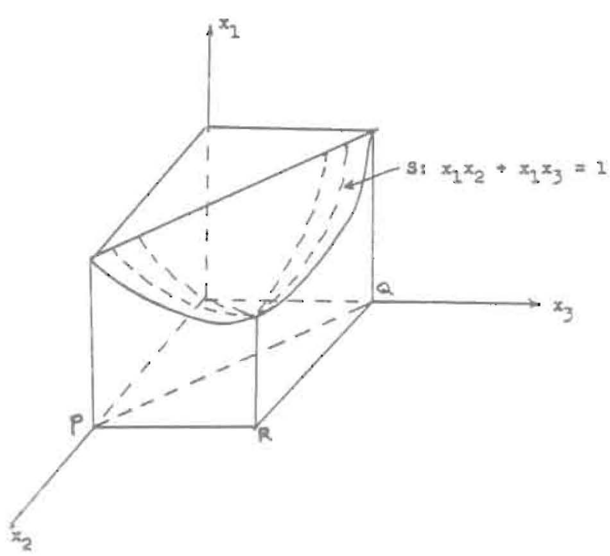


Fig. II

together with that much of the other prism wich lies below the surface S and wich rests upon triangle PRQ. And we have

$$\iiint_T x_1x_2^2x_3^2\, dV = \int_0^1 x_3^2\, dx_3 \int_0^{1-x_3} x_2^2\, dx_2 \int_0^1 x_1 dx_1$$

$$+ \int_0^1 x_3^2\, dx_3 \int_{1-x_3}^1 x_2^2\, dx_2 \int_0^{1/(x_2+x_3)} x_1\, dx_1 \qquad \text{...Eq. 1}$$

Now using the preceding example as an analogy, we seek to evaluate

$$\iiiint_R x_1x_2^2x_3^2x_4^2\, dW$$

where dW denotes the rectangular element in Cartesion 4-space and where R denotes the region of the 4-dimensional hyperspace defined by

$$R: \begin{cases} 0 \leq x_1x_2 + x_3x_4 \leq 1 \\ 0 \leq x_k \leq 1, \; k = 1,2,3,4 \end{cases}$$

For convenience let us denote the hypercube $0 \leq x_k \leq 1$ by H. And now, guided by the 3-dimensional example, we proceed to determine what portions of H constitute the region R.

The hypersurface $x_2 + x_3x_4 = 1$, obtained by putting $x_1 = 1$, is analogous to the plane $x_2 + x_3 = 1$ which divided the cube C in the 3-space problem into two prisms. Similarly, the hypersurface $x_2 + x_3x_4 = 1$ divides the hypercube H into two parts. Let us see how much of each part belongs to R. Incidentally, we can actually picture $x_2 + x_3x_4 = 1$ in a 3-space diagram as in Fig. III.

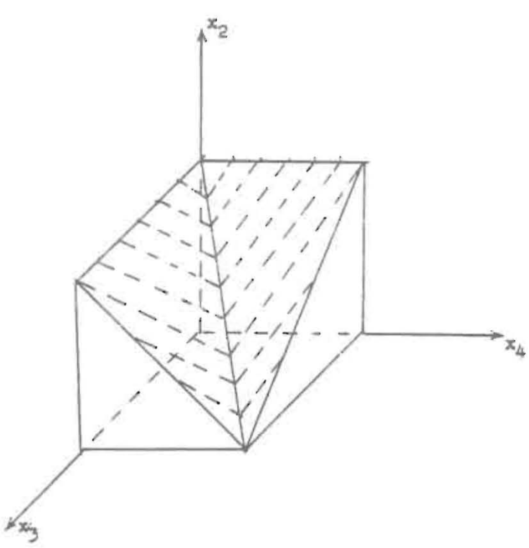


Fig. III

working with arguments similar to those to obtain the prism in Fig. I as part of region T, we conclude that <u>one portion</u> of R is made up of those points of H for wich $0 \leq x_2 + x_3x_4 \leq 1$, $0 \leq x_1 \leq 1$.

Next for the points of H where we have $x_2 + x_3x_4$ greater than unity, we must take $x_1$ sufficiently less than unity as will satisfy $x_1x_2 + x_3x_4 \leq 1$. The question is: How much less than unity? The immediate answer is $x_1$ can have any value for which $0 \leq x_1 \leq (1 - x_3x_4)/x_2$.

In the preceding 3-space problem the last integral to be evaluated (farthest to the left) in each of the two iterated integrals in Eq. 1 had zero to unity as its limits of integration. Similarly, the last two integrals in the problem before us and also the next to last will have both zero to unity as limits of integration. And so we have

$$\iiiint_R x_1x_2^2x_3^2x_4^2\, dW = \int_0^1 x_4^2\, dx_4 \int_0^1 x_3^2\, dx_3$$

$$\int_0^{1-x_3x_4} x_2^2\, dx_2 \int_0^1 x_1\, dx_1 + \int_0^1 x_4^2\, dx_4$$

$$\int_0^1 x_3^2\, dx_3 \int_{1-x_3x_4}^1 x_2^2\, dx_2 \int_0^{(1-x_3x_4)/x_2} x_1 dx_1 \qquad \text{...Eq. 2}$$

When evaluating $\iiiint x_1x_2^2x_3^2x_4^2\, dx_1dx_2dx_3dx_4$ the inner integral is evaluated first. Thus after integrating with respect to $x_1$ in the first of the two iterated integrals on the right hand side of Eq. 2 we get

$$\iiint \frac{1}{2} x_2^2x_3^2x_4^2\, dx_2dx_3dx_4$$

The second integration with respect to $x_2$ yields

$$\iint \frac{1}{2} \cdot \frac{1}{3} (1 - x_3x_4)^3\, x_3^2x_4^2\, dx_3dx_4$$

After removing the parenthesis and integrating with respect to $x_3$ and $x_4$ with limits zero and unity of integration in each case, we obtain (1/2)(1/3)(1/9 - 3/16 + 3/25 - 1/36)= 19/7200.

The second set of iterated integrals in Eq. 2 yields 37/7200. Finally, (19 + 37) / 7200= 7/900 = =.007777... . It seems as if there is no way of verifying the above result by classical analysis. An excellent project for numerical analyst to work on is the vefification of the result by a computer program.

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# A BASIC ANALOGUE OF FOX'S H-FUNCTION

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*

## ABSTRATC

Certain basic integrals and integral representations for a basic analogue of Fox's H-function are investigated in this paper.

## RESUMEN

En este trabajo estudiamos un análogo básico de la función H- de Fox. Obtenemos algunas integrales y representaciones integrales de la función.

## INTRODUCTION

The object of this paper is to define a basic analogue of Fox's H-function and to establish some of its fundamental properties. The results proved are of general character and include, as special cases, the results given earlier by Agarwal.

## DEFINITION OF A BASIC FOX'S $H_q$-FUNCTION

We define a basic analogue of Fox's H-function as

$$H_{A,\,B}^{m_1,n_1}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)\pi z^s\,ds}{\prod_{j=m_1+1}^{B} G(1-b_j+\alpha_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)G(1-s)\sin\pi s}$$

where $0 \leqslant m_1 \leqslant B$, $0 \leqslant n_1 \leqslant A$; $\alpha_i$'s and $\beta_j$'s are all positive, the contour C is a line parallel to RI ($\omega s$), with indentations if necessary, in such a manner that all poles of $G(b_j-\beta_j s)$, $1 \leqslant j \leqslant m_1$ are to the right, and those of $G(1-a_j+\alpha_j s)$, $1 \leqslant j \leqslant m_1$, to the left of C. The integral converges if $Rl[s\log(z) - \log\sin\pi s]<0$ for large values of $|s|$ on the contour i.e., if $|\{\arg(z)-\omega_2\omega_1^{-1}\log|z|\}|<\pi$.

If we set $\alpha_j = \beta_i = 1$, $1\leqslant j\leqslant A$, $1\leqslant i\leqslant B$ in (2.1), then it reduces to the basic analogue of Meijer's G-function, namely

$$G_{A,\,B}^{m_1,n_1}\left[z;q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] = H_{A,\,B}^{m_1,n_1}\left[z;q\,\middle|\,\begin{matrix}(a,1)\\(b,1)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-s)\prod_{j=1}^{n_1} G(1-a_j+s)\pi z^s\,ds}{\prod_{j=m_1+1}^{B} G(1-b_j+s)\prod_{j=n_1+1}^{A} G(a_j-s)G(1-s)\sin\pi s} \tag{2.2}$$

If we put $m_1 = B$, $n_1 = 0$; $\beta_j = \alpha_j = 1$, $1< i <A$, $1\leqslant j\leqslant B$; in (2.1) then it reduces to an $E_q$-function due to Agarwal [1], which itself is a generalization of basic analogue of MacRobert's $E_q$-function given earlier by Agarwal [2].

From the definition (2.1), it readily follows that

$$z^\sigma H_{A,\,B}^{m_1,n_1}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = (-1)^\sigma H_{A+1,B+1}^{m_1+1,n_1}\left[z;q\,\middle|\,\begin{matrix}(a_j+\alpha_j\sigma,\alpha j),(1+\sigma,1)\\(1,1),\ (b_j+\beta_j\sigma,\beta_j)\end{matrix}\right] \tag{2.3}$$

and

$$H_{A,\,B}^{m_1,n_1}\left[z^{-1};q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = -H_{B,\,A+2}^{n_1+1,m_1}\left[z;q\,\middle|\,\begin{matrix}(1-b_j,\beta_j)\\(1,1),(1-a_j,\alpha_j),(0,1)\end{matrix}\right] \tag{2.4}$$

## CERTAIN BASIC INTEGRALS INVOLVING $H_q$-FUNCTION

The following basic integrals are to be establish here.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx) H^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x) = H^{m_1+1,n_1}_{A,\,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha)\\(\sigma,\rho),(b,\beta)\end{matrix}\right], \qquad (3.1)$$

for $Rl(\sigma) > 0$ and $Rl(\rho) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, H^{m_1,n_1}_{A,\,B}\left[zx^{-u};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x)$$

$$= G(\rho-\sigma) H^{m_1+1,n_1}_{A+1,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha),\ (\sigma,u)\\(\rho,u),\ (b,\beta)\end{matrix}\right], \qquad (3.2)$$

for $Rl(\sigma) > 0$, $Rl(\rho) > 0$, $Rl(u) > 0$, $Rl(\rho-\sigma) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$.

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma} H^{m_1,n_1}_{A,\,B}\left[zx^{\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] dx$$

$$= G(1) H^{m_1,n_1}_{A+1,B}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha),\ (\sigma,\rho)\\(b,\beta)\end{matrix}\right], \qquad (3.3)$$

where the path of integration C encircles the null-point and also in the usual manner, can be deformed into a loop parallel to the imaginary axis.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx) H^{m_1,n_1}_{A,\,B}\left[zx^{\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x)$$

$$= H^{m_1,n_1+1}_{A+1,\,B}\left[z;q \,\middle|\, \begin{matrix}(1-\sigma,\rho),(a,\alpha)\\(b,\beta)\end{matrix}\right], \qquad (3.4)$$

for $Rl(\sigma) > 0$, $Rl(\rho) > 0$ and $|\arg z - \omega_2\omega_1^{-1}\log|z|| < \pi$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, H^{m_1,n_1}_{A,\,B}\left[zx^{u};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x) =$$

$$G(\rho-\sigma) H^{m_1,n_1+1}_{A+1,B+1}\left[z;q \,\middle|\, \begin{matrix}(1-\rho,u,(1,\alpha)\\(b,\beta),(1-\sigma,u)\end{matrix}\right], \qquad (3.5)$$

for $Rl(\sigma) > 0$, $Rl(u) > 0$, $Rl(\rho-\sigma) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$.

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma} H^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] dx$$

$$= G(1) H^{m_1,n_1}_{A,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta),\ (1-\sigma,\rho)\end{matrix}\right] \qquad (3.6)$$

Proof: In view of the definition (2.1), the expression on the left of (3.1), can be written as

$$\frac{G(1)}{(1-q)}\int_0^1 x^{\sigma-1} E_q(qx)\ \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)}{\prod_{j=m_1+1}^{B} G(1-b_j+\beta_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)}$$

$$\times \frac{\pi z^s x^{-\rho s}\,ds}{G(1-s)\sin \pi s}\ d(q,x)$$

If we interchange the order of integration, which is valid for $Rl(\sigma) > 0$, and $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$, then the above expression reduces to

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)\,\pi 3^s}{\prod_{j=m_1+1}^{B} G(1-b_j+\beta_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)G(1-s)\sin \pi s}$$

$$\times\left\{\frac{G(1)}{1-q}\int_0^1 x^{\rho-\sigma s-1} E_q(qx)d(q,x)\right\} ds.$$

On evaluating the inner integral with help of an integral due to Hahn [3] and interpreting the result thus obtained with the help of (2.1), we arrive at the desired result (3.1)

The remaining integrals (3.2) to (3.6) can be evaluated in the same way by making use of the results (3.16) and § 9(b) given earlier by Hahn ([3]).

## CERTAIN INTEGRALS INVOLVING BASIC MEIJER'S G-FUNCTION

If we specialize the parameters in (3.1) to (3,6) and make use of the result (2.2), the following results are obtained.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx)\, G^{m_1,n_1}_{A,\,B}\left[zx^{-u};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]d(q,x)$$

$$= \frac{G(\sigma)}{\prod_{i=1}^{u} G\left(q'^{\frac{\sigma+i-1}{u}}\right)}\, G'^{m_1+u,n_1}_{A,B+u}\left[z;q\left|\begin{matrix}(a)\\(c),(b)\end{matrix}\right.\right], \qquad (4.1)$$

where $q' = q^u$, u is a positive integer, $Rl(\sigma) > 0$, $c_i = q'^{\frac{\sigma+i-1}{u}}$ for $i = 1,\ldots,u$, $\left|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}\right| < \pi$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, G^{m_1,n_1}_{A,\,B}\left[zx^{-u};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]d(q,x)$$

$$= \frac{G(\rho-\sigma)G(\sigma)}{G(\rho)}\prod_{i=1}^{u}\frac{G\left(q'^{\frac{\rho+i-1}{u}}\right)}{G\left(q'^{\frac{\sigma+i-1}{u}}\right)}$$

$$G'^{m_1+u,n_1}_{A+u,B+u}\left[z;q\left|\begin{matrix}(a),(d)\\(c),(b)\end{matrix}\right.\right], \qquad (4.2)$$

where $q' = q^u$, u is a positive integer, $Rl(\rho) > Rl(\sigma) > 0$, $d_i = q'^{\frac{\sigma+i-1}{u}}$, $c_i = q'^{\frac{\rho+i-1}{u}}$ for $i = 1,\ldots,u$, $\left|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}\right| < \pi$.

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma}\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]dx$$

$$= \frac{G(1)\prod_{i=1}^{u} G\left(q'^{\frac{\delta+i-1}{u}}\right)}{G(\delta)}\, G'^{m_1,n_1}_{A+u,B}\left[z;q\left|\begin{matrix}(a),(c)\\(b)\end{matrix}\right.\right], \qquad (4.3)$$

where the path of integration is the same as in the case of the integral (3.3), $Rl(\sigma) > 0$, $c_i = q'^{\frac{\sigma+i-1}{u}}$, $i = 1,\ldots,u$; $q' = q^u$.

$$\frac{G(1)}{(1-q)}\int_0^1 x^{\sigma-1} E_q(qx)\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]d(q,x)$$

$$= \frac{\prod_{i=1}^{u} G\left(q'^{\frac{i-\sigma}{u}}\right)}{G(1-\sigma)}\, G'^{m_1,n_1+u}_{A+u,B}\left[z;q\left|\begin{matrix}(c),(a)\\(b)\end{matrix}\right.\right], \qquad (4.4)$$

where $q' = q^u$, u is a positive integer, $Rl(\sigma) > 0$, $c_i = q^{\frac{i-\sigma}{u}}$ for $i = 1,\ldots,u$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]d(q,x)$$

$$= \frac{G(\rho-\sigma)G(1-\sigma)}{G(1-\rho)}\prod_{i=1}^{u}\frac{G\left(q'^{\frac{-\rho+i}{u}}\right)}{G\left(q'^{\frac{-\sigma+i}{u}}\right)}$$

$$G'^{m_1+u,n_1}_{A+u,B+u}\left[z;q\left|\begin{matrix}(a),(d)\\(c),(b)\end{matrix}\right.\right], \qquad (4.5)$$

where $q' = q^u$, u is a positive integer, $Rl(\rho) > Rl(\sigma) > 0$, $d_i = q^{\frac{i-\sigma}{u}}$, $c_i = q^{\frac{i-\rho}{u}}$, for $i = 1,\ldots,u$.

$$\frac{1}{2\pi i}\int_C e_q(x)X^{-\sigma}\, G^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right]dx$$

$$= \frac{G(1)G(1-\sigma)}{\prod_{i=1}^{\rho} G\left(q'^{\frac{i-\sigma}{\rho}}\right)}\, G'^{m_1,n_1}_{A,B+\rho}\left[z;q\left|\begin{matrix}(a)\\(b),(c)\end{matrix}\right.\right], \qquad (4.6)$$

where $q' = q^\rho$, $\rho$ is a positive integer, $Rl(\sigma) > 0$, $c_i = q^{\frac{i-\sigma}{\rho}}$, for $i = 1,\ldots,\rho$.

It is interesting to observe that if we take $m_1 = B$, $n_1 = 0$ in (4.1), (4.2) and (4.3); and use the relation

$$G^{B,0}_{A,B}\left[z;q\left|\begin{matrix}(a)\\(b)\end{matrix}\right.\right] = E_q[B;b_p:A;a_t:z] \qquad (4.7)$$

then (4.1), (4.2) and (4.3) respectively give rise to the results (6.1) - (6.3) due to Agarwal [1].

## INTEGRAL REPRESENTATION FOR BASIC FOX'S $H_q$-FUNCTION

$$H^{m_1,n_1}_{A,\,B}\left[z,q\left|\begin{matrix}(a,\alpha)^*\\(b_1,1),(b_2,1),(b,\beta)^*\end{matrix}\right.\right]$$

$$= G(b_1)\prod_{j=m_1+1}^{B}\frac{1}{G(1)2\pi i}\int_{C_j} e_q(x_j)$$

$$X\, x_j^{-b_j}\,dx_j\Big\}\prod_{j=n_1+1}^{A}\left\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\,d\lambda_j\right\}$$

$$\prod_{j=1}^{n_1}\left\{\frac{G(1)}{(1-q)}\int_0^1 \lambda_j^{a_j-1}\ X\ E_q(q\lambda_j)d(q,\lambda_j)\right. \prod_{j=3}^{m_1}$$

$$\left\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}\,d(q,u_j)\right\}\frac{1}{1-q}\int_0^1 u_2^{b_2-1} \times E_q(qu_2)\,{}_1\phi_0\left(\begin{matrix}b_1;\\ -;\end{matrix}\; -u_2\prod_{j=1}^{A}\lambda_j^{-\alpha_j}\prod_{j=3}^{B}u_j^{\beta_j}/z\right)d(q,u_2), \qquad (5.1)$$

where $(a,\alpha)^*$ and $(b,\beta)^*$ denote the sequence of A and B-2 pairs $(1-a_1,\alpha_1),\ldots,(1-a_{n_1},\alpha_{n_1}),(a_{n_1+1},\alpha_{n_1+1}),\ldots,(a_A,\alpha_A);(b_3,\beta_3),\ldots,(b_{m_1},\beta_{m_1}),(1-b_{m_1+1},\beta_{m_1+1}),\ldots,(1-b_B,\beta_B)$; $Rl(b_j)>0$, $2\leqslant j\leqslant B$; $Rl(a_j)>0$, $1\leqslant j\leqslant A$.

$$H^{m,n}_{m+n-2,m+n}\left[z,q\left|\begin{matrix}(1-a_1,\alpha_1),\ldots,(1-a_n,\alpha_n),(c_1,\beta_1),\ldots,(c_{m-2},\beta_{m-2})\\ (b_1,\beta_1),\ldots,(b_m,\beta_m),(1-d_1,\alpha_1),\ldots,(1-d_n,\alpha_n)\end{matrix}\right.\right] = G(b_m)\prod_{j=1}^{n}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}(1-q\lambda_j)_{d_j-a_j-1}\,d(q,\lambda_j)/G(d_j-a_j)\right\}\prod_{k=1}^{m-2}\left\{\frac{G(1)}{1-q}\int_0^1 u_k^{b_k-1}(1-qu_k)_{c_k-b_k-1}\,d(q,u_k)/G(c_k-b_k)\right\}\frac{1}{1-q}\int_0^1 u_{m-1}^{b_{m-1}-1}E_q(qu_{m-1})\,{}_1\phi_0\left(\begin{matrix}b_m;\\ -;\end{matrix}\; -\frac{z}{u_{m-1}}\prod_{j=1}^{n}\lambda_j^{\alpha_j}\prod_{j=1}^{m-2}u_j^{-\beta_j}\right)d(q,u_{m-1}), \qquad (5.2)$$

where $Rl(d_j)>Rl(a_j)>0$, $Rl(c_k)>Rl(b_k)>0$, $Rl(\alpha_j)>0$, $\beta_m=\beta_{m-1}=1$, $Rl(\beta_k)>0$, $1\leqslant j\leqslant n$; $1\leqslant k\leqslant m-2$; α's and β's are all positive quantities.

$$H^{m_1,A-m_1}_{A-2,\,B}\left[z;q\left|\begin{matrix}(1-a_{m+1},\alpha_{m+1}),\ldots,(1-a_A,\alpha_A),\ (a_3,\beta_3),\ldots,(a_m,\beta_m)\\ (b_1,1),(b_2,1),(b_3,\alpha_3),\ldots,(b_m,\alpha_m),(1-b_{m_1+1},\beta_{m+1}),\ldots,(1-b_B,\beta_B)\end{matrix}\right.\right] = G(b_1)\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{k=3}^{m_1}\frac{G(1)}{1-q}\int_0^1\lambda_k^{b_k-1}(1-q\lambda_k)_{a_k-b_k-1}\,d(q,\lambda_k)/G(a_k-b_k)\Big\}\prod_{t=m+1}^{A}\frac{G(1)}{1-q}\int_0^1 x_t^{a_t-1}E_q(qx_t)\,d(q,x_t)\Big\}\frac{1}{1-q}\int_0^1\lambda_2^{b_2-1}E_q(q\lambda_2)\,{}_1\phi_0\left(\begin{matrix}b_1;\\ -;\end{matrix}\; -z\prod_{j=m_1+1}^{B}u_j^{-\beta_j}\prod_{K=3}^{m_1}\lambda_k^{\beta_k}\prod_{t=m_1+1}^{A}x_t^{\alpha_t}/\lambda_2\right)d(q,\lambda_2) \qquad (5.3)$$

where $Rl(a_j)>0$, $Rl(a_k)>Rl(b_k)>0$, $3\leqslant k\leqslant m_1$, $m_1+1\leqslant j\leqslant A$, $Rl(b_t)>0$, $m_1+1\leqslant t\leqslant B$; α's and β's are all positive quantities.

Proof: To prove (5.1) we start with the known result ( 2 ; (2.1))

$$E_q(a,b::z)=\frac{G(a)}{1-q}\int_0^1 E_q(q\lambda)\lambda^{b-1}\,{}_1\phi_0(a;-\lambda/z)\,d(q,\lambda)$$

$$=\frac{1}{2\pi i}\int_C\frac{G(a-s)G(b-s)\pi z^s\,ds}{G(1-s)\sin\pi s},$$

where the contour C is a line parallel to $Rl(\omega s)=0$. The integral converges if $Rl[s\log(z)-\log\sin\pi s]<0$ for large values of $|s|$ on the contour, i.e., if $|\{\arg(z)-\omega_2\omega_1^{-1}\log|z|\}|<\pi$.

The expression of the right of (5.1) can be written as

$$\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\frac{1}{G(1)2\pi i}\int_{C_j}e_q(\lambda_j)\lambda_j^{-a_j}d\lambda_j\Big\}\prod_{j=1}^{n_1}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}E_q(q\lambda_j)\,d(q,\lambda_j)\right\}$$

$$\prod_{j=3}^{m_1}\left\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}d(q,u_j)\right\}\frac{1}{2\pi i}\int_C\frac{G(b_1-s)}{G(1-s)}\,\frac{G(b_2-s)\pi z^s}{\sin\pi s}\prod_{j=3}^{B}u_j^{-\beta_j s}\prod_{j=1}^{A}\lambda_j^{\alpha_j s}ds = \prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\frac{1}{G(1)2\pi i}\left\{\int_{C_j}e_q(\lambda_j)\lambda_j^{-a_j}d\lambda_j\right\}$$

$$\prod_{j=1}^{n_1}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}E_q(q\lambda_j)\,d(q,\lambda_j)\right\}\prod_{j=4}^{m_1}\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}d(q,u_j)\Big\}\frac{G(1)}{1-q_A}\int_0^1 E_q(qu_3)u_3^{b_3-1}d(q,u_3)\,\frac{1}{2\pi i}\int_C\frac{G(b_1-s)G(b_2-s)\pi z^s\prod_{j=1}^{A}\lambda_j^{\alpha_j s}\,ds}{G(1-s)\sin\pi s\prod_{j=3}^{B}u_j^{\beta_j s}}$$

On changing the order of integration which is valid by absolute convergence of both the integrals, for $Rl(b_3)>0$ and $|\{\arg z-\omega_2\omega_1^{-1}\log|z|\}|<\pi$; the R.H.S. for the above expression becomes

$$\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\left\{\frac{1}{G(1)2\pi i}\right.$$

$$\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\, d\lambda_j \Big\} \prod_{j=1}^{n_1} \Big\{ \frac{G(1)}{1-q} \int_0^1 \lambda_j^{a_j-1} E_q(q\lambda_j)\, d(q,\lambda_j)\Big\} \prod_{j=4}^{m_1} \Big\{ \frac{G(1)}{1-q} \int_0^1 E_q(qu_j) u_j^{b_j-1}\, d(q,u_j)\Big\} \cdot \frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)\pi z^s \prod_{j=1}^{A} \lambda_j^{\alpha_j s}}{G(1-s)\sin \pi s \prod_{j=4}^{B} u_j^{\beta_j s}} \Big\{ \frac{G(1)}{1-q} \int_0^1 E_q(qu_3) u_3^{b_3-\beta_3 s-1}\, d(q,u_3)\Big\}\, ds$$

On evaluating the innermost integral, it gives

$$\prod_{j=m_1+1}^{B} \Big\{ \frac{1}{G(1)2\pi i} \int_{C_j} e_q(u_j) u_j^{-b_j}\, du_j \Big\} \prod_{j=n_1+1}^{A} \Big\{ \frac{1}{G(1)2\pi i} \int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\, d\lambda_j \Big\} \prod_{j=1}^{n_1} \Big\{ \frac{G(1)}{1-q} \int_0^1 \lambda_j^{a_j-1} E_q(q\lambda_j)\, d(q,\lambda_j)\Big\} \prod_{j=4}^{m_1} \Big\{ \frac{G(1)}{1-q} \int_0^1 E_q(qu_j) u_j^{b_j-1}\, d(q,u_j)\Big\} \frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)G(b_3-\beta_3 s)\pi z^s \prod_{j=1}^{A} \lambda_j^{\alpha_j s}}{G(1-s)\sin \pi s \prod_{j=4}^{B} u_j^{\beta_j s}}\, ds.$$

Similarly on using the known integral due to Hahn [3], $(m_1-3)$-times and $n_1$-times, the above expression reduces to

$$\prod_{j=m_1+1}^{B} \Big\{ \frac{1}{G(1)2\pi i} \int_{C_j} e_q(u_j) u_j^{-b_j}\, du_j \Big\} \prod_{j=n_1+1}^{A} \Big\{ \frac{1}{G(1)2\pi i} \int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\, d\lambda_j \Big\} \frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s) \prod_{j=3}^{m_1} G(b_j-\beta_j s) \prod_{j=1}^{n_1} G(a_j+\alpha_j s) \prod_{j=n_1+1}^{A} \lambda_j^{\alpha_j s}\, \pi z^s}{G(1-s)\sin \pi s \prod_{j=m_1+1}^{B} u_j^{\beta_j s}}\, ds$$

Now again, changing the order of integration, which is justified as before, and using the known result due to Hahn ( 3 , §9(b)), $(A-n_1)$-times and $(B-m_1)$-times to evaluate the inner integrals, we finally obtain

$$H^{m_1,n_1}_{A,B}\left[ z;q \,\middle|\, \begin{matrix} (1-a_1,\alpha_1),\dots,(1-a_{n_1},\alpha_{n_1}),(a_{n_1+1},\alpha_{n_1+1}),\dots,(a_A,\alpha_A) \\ (b_1,1)(b_2,1)(b_3,\beta_3)\dots(b_{m_1},\beta_{m_1})\,(1-b_{m_1+1},\beta_{m_1+1})\dots(1-b_B,\beta_B) \end{matrix} \right] = \frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)\prod_{j=3}^{m_1} G(b_j-\beta_j s) \prod_{j=1}^{n_1} G(a_j+\alpha_j s)\pi z^s\, ds}{\prod_{j=m_1+1}^{B} G(b_j+\beta_j s) \prod_{j=n_1+1}^{A} G(a_j-\alpha_j s) G(1-s)\sin \pi s}, \qquad (5.4)$$

which converges for $|\{\arg z - \omega_2\omega_1^{-1} \log|z|\}| < \pi$.

The results (5.2) and (5.3) can be established in the same way if we employ the results (3.16)and § 9(b) given by Hahn [3], to evaluate the inner integrals.

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