ABSTRACT

A decomposition of $\mathbb{C}^n$ into a finite direct sum of orthogonal subspaces can be conveniently represented by its orthogonal projector frame, which is the collection of the corresponding orthogonal projectors. Two such decompositions whose frames are close are shown to be linearly homeomorphic and homotopic. In a recent work we compared the resulting geodesic arcs with naturally arising paths resulting from interpolating the balanced transformation, and found them cubically close. In this work we describe an efficient algorithm to compute the balanced transformation.

RESUMEN

Una descomposición de $\mathbb{C}^n$ en una suma directa finita de subespacios ortogonales puede ser representada convenientemente por su cuadro proyector ortogonal, la cual es la colección de los proyectores ortogonales correspondientes. Dos de tales descomposiciones, cuyos cuadros son cerrados, son "homórfico y homotópico". En este trabajo, se describe un algoritmo eficiente para computar la transformación balanceada.

2. NOTATIONS AND PRELIMINARIES

By an orthogonal $r$-frame $E$ on $\mathbb{C}^n$, we mean $E = (E_1, E_2, \ldots, E_r)$ where $E_j \in \mathbb{C}^{n \times n}$, $1 \leq j \leq r$, satisfy

\[
\begin{align*}
(1) \quad & 0 = E_j^2 = E_j^* \\
(2) \quad & \sum_{j=1}^{r} E_j = I
\end{align*}
\]

Clearly equation (2.1) implies that the $E_j$'s are pairwise disjoint, since if

\[
x \in \mathbb{R}(E), \quad |x| = \sum_{j=1}^{r} |E_j x|^2 \quad \text{so} \quad \sum_{j=1}^{r} |E_j x|^2 = 0.
\]
Two frames $E$ and $F$ are said to be unitarily similar if there exists a unitary matrix $V$ such that $VE = VF$, that is, $VE_j = FV_j$, $1 \leq j \leq r$. The unitary similarity orbit of a fixed frame $E$, denoted by $\mathcal{S}(E)$, is the set of frames which are unitarily similar to $E$, namely

$$\mathcal{S}(E) = \{ VEV^*, V \text{ is unitary matrix} \}. \quad (2.2)$$

In [7] the set $\mathcal{S}(E)$ is studied where it is shown to be a Riemannian manifold. In fact if $F$ is a close frame to $E$, then certainly $F \in \mathcal{S}(E)$. This will be the case if for example

$$[E-F^\dagger]_{1 \leq j \leq s} \leq 1.$$  

A particular unitary $U$ which realizes the equivalence of the frames $E$ and $F$ is

$$U = U(F,E) = \left( \sum_{j=1}^{r} F_j E_j + \sum_{j=1}^{r} E_j F_j \right)^{1/2}. \quad (2.3)$$

It can be easily checked that $U(F,E)E = F U(F,E)$, so $U(F,E)$ maps the subspace $\mathcal{R}(E)$ onto $\mathcal{R}(F)$, $\mathcal{S}(F)$. We also note that $U(F,E)^* = U(E,F)$; for this reason we call it the balanced transformation.

If we want to move the frame $E$ onto $F$ in the most natural and efficient way within the set of $r$-frames on $C^n$, this will not be achieved by considering the straight line segment. This is because the straight line segment does not lie in $\mathcal{S}(E)$, since if

$$t \rightarrow E + t(F-E), \text{ Otsl} \text{ lies in } \mathcal{S}(E),$$

then

$$F_j(t) = E_j + t(F_j-E_j)(1 \leq j \leq r, \; 0 \leq t \leq 1)$$

is an orthogonal projector. Thus

$$0 = F_j(t) - F_j = (t^2 - t)(F_j-E_j) \quad 0 \leq t \leq 1, \; 1 \leq j \leq r.$$  

This implies that $E_j = F_j$ for all $j$ and hence $E = F$.

However, a locally minimal arc in $\mathcal{S}(E)$ which connects $E$, $F$ will be the geodesic arc $t \rightarrow r(t) = (F(t)^{-\dagger}F(t)^{-1})^{1/2}$, $t \in [0,1]$. $F(0) = E$, $F(1) = F$, where $F_j(t)$ is defined by

$$F_j(t) = \exp(tL) E_j \exp(-tL), \quad '1 \leq j \leq r. \quad (2.4)$$

Here $L$ is a skew hermitian matrix ($L = -L^*$) and satisfies the matrix equation

$$r \sum_{j=1}^{r} F_j \exp(tL) E_j \sum_{j=1}^{r} E_j L E_j = 0 \quad (2.5)$$

It is shown in [7] that the length of the geodesic arc connecting $E$ and $F$ is $\|L\|_F^2 (\|L\|_F^2 = (tr L^2)^{1/2})$, which justifies calling $\exp L$ the direct rotation between $E$ and $F$.

Both unitaries $\exp L$ and $U(F,E)$ give rise to paths in $\mathcal{S}(E)$ connecting $E$ and $F$. However these paths are in general different [10]. The first unitary has geometric significance. The second unitary $U(F,E)$ is not the most natural way to move the subspace $\mathcal{R}(E)$ onto $\mathcal{R}(F)$, $\mathcal{S}(F)$, but still has the advantage that it is expressed algebraically in terms of $E$ and $F$. Also it is recently shown in [10] that $U(F,E)$ is still close to $\exp L$, namely

$$\|U(F,E) - \exp L\| = O(\|F-E\|).$$

So even if one is interested in computing $\exp L$ via solving (2.5) iteratively, a good initial approximation will be $U(F,E)$.

Let $X = \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_r$ and $M_1 \otimes M_2 \otimes \cdots \otimes M_r$ be the decomposition of $C^n$ arising from $E$ and $F$ respectively. That is, $X = X(E)$ and $M_j = F_j E_j$, $1 \leq j \leq r$. There are different ways to identify subspaces of $C^n$. In our case we will define the subspaces using orthonormal matrices.

Namely, for $j = 1,2,\ldots,r$, let $V_j, W_j \in C^{n_j}$ where

$$\left\{ \begin{array}{c} V_j V_j^* = I \\ W_j W_j^* = E_j \\ W_j V_j = I \\ V_j W_j = F_j \end{array} \right. \quad (2.6)$$

The above identification is unique only within a post-multiplication by an arbitrary unitary $n_j \times n_j$ matrix.

The balanced transformation $U(F,E)$ can be computed directly using equation (2.3), where the inverse square root of an $n \times n$ matrix is to be computed. Such an inverse square root can be computed using, for example, the numerically stable technique suggested in [14]. However, if $n$ is large, the above procedure which is of order $O(n^3)$ will be computationally expensive. The purpose of the next section is to propose a factorization of $U(F,E)$ so that only lower order matrices, $n \times n_j$ and $n_j \times n_j$ will enter the calculations. The saving will be remarkable, when the restriction of $U(F,E)$ to $E_i$ is required with $n_i \leq cn$.

Remark 2.1. If the subspaces $E_i, M_j$ are defined by $A_i, B_j$ respectively, $1 \leq i, j \leq r$, then a QR factorization step is needed to get $V_j, W_j$. 

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3. A FACTORIZATION OF $U(F,E)$

The following relations are well known, cf. [3, 8], we list them for the sake of completeness. For two frames $E$ and $F$ we define

$$C_j = (F_j + E_j - I)^2,$$  

then we have for $1 \leq j \leq r$:

$$(i) \ 0 \leq C_j \leq I$$

$$(ii) \ C_j E_j = E_j C_j, \ C_j F_j = F_j C_j$$  \hspace{1cm} (3.2)

$$(iii) \ E_j C_j E_j = E_j F_j E_j, \ F_j C_j F_j = F_j E_j F_j.$$  \hspace{1cm} (3.3)

Equation (2.3) can be equivalently written as

$$U(F,E) = \sum_{j \in \Delta} F_j E_j F_j E_j (E_j F_j E_j)^{-1/2}$$

$$= \left( \sum_{j \in \Delta} F_j E_j F_j E_j \right) \left( \sum_{j \in \Delta} E_j F_j E_j F_j E_j \right)^{-1/2}$$

$$= \sum_{j \in \Delta} F_j C_j^{-1/2} E_j.$$  \hspace{1cm} (3.4)

This follows by direct calculations, using properties of the $C_j$'s listed in (3.2). Further, set

$$T_j = T(F_j, E_j) = C_j^{-1/2} (F_j + E_j - I), \ 1 \leq j \leq r$$

and associate $Z_j$ with $T_j$ where

$$Z_j = \frac{1}{2} (I + T_j), \ 1 \leq j \leq r.$$  \hspace{1cm} (3.5)

The following theorem records some properties of the $T_j$'s. Also it expresses $U(F,E)$ in terms of the $T_j$'s.

**Theorem 3.1.** Each $T_j$ is a hermitian involutory matrix exchanging $E_j$ with $M_j$ and

$$E_j T_j E_j = 0, \ F_j T_j F_j = 0, \ 1 \leq j \leq r.$$  \hspace{1cm} (3.6)

The balanced transformation $U(F,E)$ can be expressed in terms of the $T_j$'s as follows

$$U(F,E) = \sum_{i \in \Delta} T_j E_j.$$  \hspace{1cm} (3.7)

Further, $Z = (Z_1, Z_2, ... , Z_r)$, where the $Z_j$'s are defined by (3.5), satisfies

$$(T(Z, E)^{T} Z)_{T}^2 = T(F, E) T.$$  \hspace{1cm} (3.8)

Here

$$T(Z, E) = \left( T_j (Z_j, E_j) \right)^{T}, \ T_0 = \left( T_0 \right)^{T}, \ T_0 = 2E_{-1}.$$

**Proof.** From equation (3.4), we have by direct calculations using (3.2), $T_j = T_j$ and $T_j = 1.

Further $T_j E_j = C_j^{-1/2} F_j E_j = F_j T_j$ since $F_j$ commutes with $C_j$. Hence indeed $T_j$ exchanges $Z_j$ and $M_j$. To prove (3.6) we note that $F_j T_j F_j = 0$ is equivalent to $E_j T_j E_j = 0$ since $F_j T_j E_j = T_j (E_j T_j) T_j$, hence we show that $E_j T_j E_j = 0$.

$$E_j T_j E_j = C_j^{-1/2} F_j E_j = C_j^{-1/2} E_j C_j E_j = C_j^{-1/2} E_j = 0,$$

since $C_j = 0$. From equation (3.3) we have

$$U(F,E) = \sum_{j \in \Delta} F_j C_j^{-1/2} E_j = \sum_{j \in \Delta} C_j^{-1/2} F_j E_j$$

$$= \sum_{j \in \Delta} C_j^{-1/2} (F_j + E_j - 1) E_j = \sum_{j \in \Delta} T_j E_j$$

hence (3.7) follows. Now since $T_j$ is a hermitian involutory matrix, then $Z_j$ is an orthogonal projector. Hence if we define $T_j (Z_j, E_j)$ by equation (3.4), $T_j (Z_j, E_j)$ will be a hermitian involution which exchanges $R(Z_j)$ with $R(E_j)$. Further, since

$$T_j (Z_j, E_j) E_j = Z_j T_j (Z_j, E_j),$$

we have

$$[T_j (Z_j, E_j) (2Z_j - 1)]^2 = T_j (Z_j, E_j) [2C_j - 1] (2Z_j - 1);$$

but $T_j^2 (Z_j, E_j) = I$, hence (3.8) follows.

**Remark 3.1.** The components $Z_j$ of $Z$ are orthoprojectors on subspaces which can be named as the bisector subspaces of $E_j$ and $M_j$, this can be seen from equation...
(3.8), see also [3] in case of a pair of subspaces.
However, in general $Z \not\in \mathcal{E}(E)$, $r > 2$; in case of a 2-frame $Z = (Z_1, Z_2)$ will be a frame. This follows since

\[ Z_1 + Z_2 = \frac{1}{2} (Z_1 + T_1 + T_2), \text{ with } T_2 = -T_1, \text{ hence indeed } Z_1 + Z_2 = L. \]

We construct an orthonormal basis of the bisector subspace $\mathcal{R}(Z_j)$ in terms of $V_j$ and $W_j$. This construction extends in some sense the calculation of the bisector of two unit vectors in the plane. Once this base is established we can compute $T_j$ and consequently $U$ can be computed via equation (3.7).

**Theorem 3.2.** Let $(V_j)_{j=1}^r$ and $(W_j)_{j=1}^r$ be as defined in (2.6).

(i) There exists an orthonormal matrix $X_j$, $1 \leq j \leq r$, such that $\mathcal{R}(X_j) = Z_j$, and $X_j$ is the closest orthonormal basis to $V_j$.

(ii) Set $Y_j = W_j V_j$, then $X_j$ in part (i) can be expressed as follows:

\[ X_j = W_j V_j (V_j V_j)^{-1/2}, \text{ for } 1 \leq j \leq r. \]

(iii) If $G_j = X_j V_j$, then $N_j = G_j (G_j V_j)^{-1/2}$, is an orthonormal basis of the bisector subspace $\mathcal{R}(Z_j)$.

**Proof.** Define $H_j$, $1 \leq j \leq r$, by

\[ H_j = W_j U(F,E)V_j. \]

Upon using equations (2.3) and (2.6) we have $H_j^* H_j = H_j H_j^* = 1$. Let

\[ X_j = W_j H_j, \]

so $X_j^* X_j = I$, and $X_j^* W_j = F_j$; and indeed $X_j$ is a basis for $\mathcal{R}(F_j)$ which is closest to $V_j$ (because $X_j = U(F,E)V_j$). To prove (ii), we have

\[ H_j = W_j U(F,E)V_j = W_j T_j V_j, \]

as follows from the factorization of $U(F,E)$ in equation (3.7). Hence

\[ H_j = W_j (E_j + F_j - I) C_j^{1/2} V_j \]

\[ = W_j (V_j V_j + W_j W_j - I) C_j^{1/2} V_j. \]

But

\[ C_j V_j = (I - F_j + E_j) V_j. \]

\[ = (I - V_j V_j^* - W_j W_j^* + V_j V_j^* W_j W_j^*) V_j \]

\[ = V_j (W_j V_j) (W_j V_j). \]

Hence if we set

\[ Y_j = W_j V_j, \]

we get

\[ C_j V_j = V_j I_j. \]

\[ C_j^2 V_j = C_j V_j = V_j I_j. \]

Inductively, $C_j^n V_j = V_j I_j^n$ for any positive integer $m$. Hence $f_c V_j = V_j I_j^n$ for any continuous function on $[0,1]$, so it is true for the inverse square root function, that is,

\[ C_j^{-1/2} V_j = V_j I_j^{-1/2}. \]

But

\[ H_j = W_j C_j^{1/2} V_j \]

\[ = W_j V_j I_j^{-1/2} = V_j (Y_j Y_j)^{-1/2}. \]

Thus

\[ X_j = W_j Y_j (Y_j Y_j)^{-1/2}, \text{ for } 1 \leq j \leq r. \]

Next we use $X_j$ to establish basis for $\mathcal{R}(Z_j)$. We set

\[ G_j = X_j + V_j. \]

Now $G_j$ is a basis for $\mathcal{R}(Z_j)$, this is because $G_j = X_j + V_j = T_j V_j + V_j = Z_j V_j$, hence $\mathcal{R}(G_j) = \mathcal{R}(Z_j)$. An orthonormal basis $N_j$ for $\mathcal{R}(G_j)$ can be established as

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That is, \( N_j \) is the unitary polar factor in the polar decomposition of \( G_j \). So indeed \( N_j^*N_j = Z_j \) and the proof is complete.

**Remark 3.2.** We note that all inverse square root operations involve matrices of lower order \( n_j < n_j \). Further, if it is only required to compute \( U(F,E) \), then factorization (3.7) reduces the problem to computing \( T_{E_k} \) only. These two points illustrate the advantage of using (3.7) to compute \( U(F,E) \) rather than the direct formula (3.3).

The system plays a prominent role. Here one is interested in bases for the bisector subspace. For example, in factor analysis, the choice of coordinate systems satisfies the desired facts we refer to [5.15,17]. In the case of two frames \( F_j \) for the principal angles or bases for bisector subspaces. These angles constitute the spectrum of a hermitian positive definite matrix \( o_j \). In fact

\[
C = \cos^2 o_j - \cos^2 o_j = 2 \cos^2 o_j - 1 = 2 \csc^2 o_j.
\]

We now summarize the computational procedure to compute \( U \), as well as other relevant quantities such as the principal angles or bases for bisector subspaces, when the orthonormal matrices \((W_j)_{j=1}^r \) are given.

**Step 1.** For \( j = 1, \ldots, r \) do Step 2 to Step 6.

**Step 2.** Set \( Y_j = W_jV_j \).

**Step 3.** Find SVD of \( Y_j \) set \( B_j = Y_j(Y_j^TW_j)^{1/2} \).

**Step 4.** Set \( X_j = W_jB_j \), \( G_j = X_j + V_j \).

There are different techniques to achieve this [6,13]. One approach is based on the use of SVD of the given matrix. Let \( A \in \mathbb{C}^{n \times n} \), \( k \neq 0 \) be a full rank matrix, consider SVD of \( A \)

\[
A = U \Sigma V^T,
\]

where \( U = \text{diag} (S_1(A), S_2(A), \ldots, S_N(A)) \) and \( S_j(A) \geq S_{j+1}(A) \).

Here \((S_j(A))_{j=1}^r\) are called the singular values of \( A \). For \( \Sigma = \text{diag} \{ \lambda_j \}_{j=1}^r \) where \( \lambda_j \in \mathbb{C}^{1 \times 1} \), then in the polar decomposition of \( A, A = BH \), the unitary polar factor \( B \) is \( Q \). Note that \( B = A(A^*A)^{1/2} \). In [6] another approach was proposed to construct the polar factor of a square matrix by applying the iteration

\[
B_{r+1} = \frac{1}{2} (B_r^*B_r)^{1/2}.
\]

Then \( B \rightarrow B \) quadratically. If the matrix \( A \) is not square a QR factorization step is needed and then we apply (4.2) to \( R \). The latter approach does not give information about singular values.

**Remark 4.1.** The algorithm to be described will enable us to compute the angles between subspaces \( F_j \) and \( M_j \). Each pair of subspaces \( F_j, M_j \), is characterized in terms of certain angles called principal angles. These angles constitute the spectrum of a hermitian positive definite matrix \( o_j \). In fact

\[
C_j = \cos^2 o_j - 1 = \cos^2 o_j = 2 \csc^2 o_j.
\]
Step 5 Compute \( N_j = G_1(G_j G_j)^{1/2} \).
Step 6 Set \( Z_j = N_j Z_j T_j = 2Z_j - I, E_j = V_j V_j^* \).
Step 7 Set \( U = \bigoplus_{j=1}^{m} U_j \).

In applying the previous algorithm, the angles between subspaces can be computed, if required, in Step 2 as pointed out in Remark 4.1. In Step 5, we can compute any orthonormal bases for \( R(G_j) \), for example a QR factorization step will be enough, however \( N_j \) is the optimal one [13]. Finally the inverse square root encountered may also be computed as in [14].

We illustrate the previous algorithm by the following numerical example.

**Example 4.1.** Consider the following subspaces in \( \mathbb{R}^3 \) determined by

\[
V_1 = e_1, \quad V_2 = [e_2, e_3], \quad V_3 = e_4
\]

and

\[
W_1 = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}
\]

Applying the previous algorithm, the principal angles between \( R(V_j) \) and \( R(W_j) \), \( 1 \leq j \leq 3 \) are

\[
\{ \pi/3, \pi/4, \pi/4, \pi/3 \}
\]

The balanced transformations is

\[
U(F,E) = \begin{bmatrix}
0.50000 & 0.00000 & -0.70711 & 0.50000 \\
-0.50000 & 0.70711 & 0.00000 & -0.50000 \\
0.50000 & 0.00000 & 0.70711 & -0.50000 \\
0.50000 & 0.70711 & 0.00000 & 0.50000 \\
\end{bmatrix}
\]

We remark that iteration (4.2) can be applied in Step 2 instead of the SVD if the principal angles are not required.

### 5. A Perturbation Inequality

Let \( E, F \) be the frames associated with the decomposition of \( C_m \) into \( \mathbb{R} \) \( Z_1 \cdots \otimes Z_n \) and \( \mathbb{C} \otimes \cdots \otimes \mathbb{C} \). Suppose these subspaces are perturbed so that we have \( \mathbb{R} \otimes Z_1 \cdots \otimes Z_n \) and \( \mathbb{C} \otimes \cdots \otimes \mathbb{C} \). As before suppose \( \hat{E}, \hat{F} \) are defined by orthonormal matrices \( \{ \hat{V}_j \}_{j=1}^m \)

and \( \{ \hat{W}_j \}_{j=1}^r \) as in (2.6).

We set for \( 1 \leq j \leq r \)

\[
\begin{cases}
C(\theta_j) = \text{diag} \{ \sigma_1, \ldots, \sigma_{j-1}, \sigma_j \}, & \sigma_j \geq \sigma_{j+1} \geq \cdots \geq \sigma_{r+1} \\
S(\theta_j) = \text{diag} \{ \mu_1, \ldots, \mu_{j-1}, \mu_j \}, & \mu_j \geq \mu_{j+1} \geq \cdots \geq \mu_{r+1} \\
\end{cases}
\]

Also we set

\[
\begin{cases}
(\mu_k^0)_{k=1}^n = (\sin \theta_k^0_{k=1}^n, \\
(\sigma_k^0)_{k=1}^n = (\cos \theta_k^0_{k=1}^n)
\end{cases}
\]

where \( (\theta_k^0)_{k=1}^n \) are the principal angles between \( \mathbb{R}^n \). A relation similar to (5.2) holds where \( (\theta_k^0)_{k=1}^n \) are the principal angles between \( \mathbb{C}^n \). The purpose of this section is to derive some perturbation inequalities for \( C(\theta_j) \) and \( S(\theta_j) \) and \( U(F,E) \) in terms of the perturbations in \( V_j \) and \( W_j \).

The perturbation bounds in this section will be cast in terms of unitarily invariant norms. A unitarily invariant norm on \( C_m \times n \) is a matrix norm with the additional property that for \( A \in C_m \times n \)

\[
\| PAQ \| = \| A \|
\]

if \( P, Q \) are unitaries. We shall be dealing with matrices of varying dimensions, hence we shall consider a family of unitarily invariant norms defined on

\[
\{ U \}_{n=1}^m C_m \times n
\]

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We refer to [15] for details about unitarily invariant norms. In particular \( \| \cdot \|_j \) will denote the spectral norm.

The following theorem is well known [12].

**Theorem 5.1.** Let \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_p \) and \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \) be the singular values of the matrices \( A, B \), then

\[
\| \text{diag}(\rho_1 - \sigma_1, \ldots, \rho_p - \sigma_p) \| = \| A - B \|
\]

in any unitarily invariant norm.

Let \( C(\Theta), S(\Theta), C(\Theta)^*, S(\Theta)^* \) be defined as in (5.1) and (5.3), then we now prove the perturbation inequalities.

**Theorem 5.2**

(i) \( \| \mathcal{E}_j - \mathcal{E}_j^* \|_2 = \min \{ \| V_j - \overline{V}_j \|, \| V_j^* - \overline{V}_j^* \| \} \)

(ii) \( \| C(\Theta) - C(\Theta)^* \| \leq \| V_j - \overline{V}_j \|, \| W_j - \overline{W}_j \| \)

(iii) \( \| S(\Theta) - S(\Theta)^* \| \leq \| V_j - \overline{V}_j \| \| W_j^* - \overline{W}_j^* \| \)

in any unitarily invariant norm.

**Proof.**

\[
\| \mathcal{E}_j - \mathcal{E}_j^* \| = \| V_j V_j^* - \overline{V}_j \overline{V}_j^* \| \leq \| V_j + \overline{V}_j \| \| V_j - \overline{V}_j \|
\]

\[
\leq 2 \| V_j - \overline{V}_j \|
\]

Similarly

\[
\| \mathcal{E}_j - \mathcal{E}_j^* \|_f = \sqrt{\| V_j - \overline{V}_j \|^2 + \| V_j^* - \overline{V}_j^* \|^2} \leq \sqrt{\| V_j - \overline{V}_j \|^2 + \| V_j^* - \overline{V}_j^* \|^2} \leq 2 \| V_j - \overline{V}_j \|.
\]

Hence (a) follows. A similar inequality holds for

\[
\| \mathcal{F}_j - \mathcal{F}_j^* \|.
\]

For part (iii), we have

\[
\| W_j^* V_j - \overline{W}_j^* \overline{V}_j \| \leq \| W_j^* V_j - \overline{W}_j^* \overline{V}_j \| + \| \overline{W}_j^* V_j - \overline{W}_j \overline{V}_j \| \leq \| W_j^* V_j - \overline{W}_j \overline{V}_j \| + \| \overline{W}_j \overline{V}_j - \overline{W}_j \overline{V}_j \|
\]

However the singular values of \( W_j^* V_j \) are precisely the diagonal elements of \( C(\Theta) \) (Remark 4.1), similarly for \( \overline{W}_j \overline{V}_j \). Now we apply Theorem 5.1 to get

\[
\| C(\Theta) - C(\Theta)^* \| \leq \| W_j^* \overline{V}_j - \overline{W}_j \overline{V}_j \| \leq \| W_j^* V_j - \overline{W}_j \overline{V}_j \| + \| V_j - \overline{V}_j \|
\]

Similarly we can prove (iii).

**Remark 5.1.** The constant in the inequality in part (i) is reduced to 1 in case of the spectral norm while it is \( \sqrt{2} \) in the Frobenius norm. This is because \( V_j^* \overline{V}_j \) has the same singular values as \( V_j \overline{V}_j \). Namely, we have

\[
\| V_j^* \overline{V}_j - \overline{V}_j \overline{V}_j \| = \| V_j - \overline{V}_j \|
\]

In particular

\[
\| \mathcal{E}_j - \mathcal{E}_j^* \|_f = \sqrt{\| V_j - \overline{V}_j \|^2 + \| V_j^* - \overline{V}_j^* \|^2} \leq \| V_j - \overline{V}_j \|
\]

Similarly

\[
\| E_j - \overline{E}_j \|_f \leq \sqrt{2} \| V_j - \overline{V}_j \|
\]

Finally we present a perturbation inequality for \( UF(\overline{E}) \). For that we need the following theorem which is also of interest.

**Theorem 5.3.** Let \( K, \overline{K} \) be skew hermitian matrices, then

\[
\| e^K - e^{\overline{K}} \| \leq \| K - \overline{K} \|
\]

in any unitarily invariant norm.

**Proof.** The proof is based on the following identity

\[
\frac{d}{dt} e^{(t-iK)x} e^{(t-iK)^* y} x = -K e^{(t-iK)x} e^{(t-iK)^* y} x + e^{(t-iK)x} e^{(t-iK)^* y} K x.
\]

This identity is introduced and used in [18]. Hence upon integration.
However $|e^{t \mathbf{K}_1} - e^{t \mathbf{K}_2}| = \int_0^t |e^{t \mathbf{K}_1} (e^{t \mathbf{K}_2 - t \mathbf{K}_1})| dt$.

In [8], $\text{U}(\mathbf{F}, \mathbf{E})$ was locally characterized, and it was shown that if $\mathbf{K} = \log \text{U}(\mathbf{F}, \mathbf{E})$, then $\mathbf{K}$ is the unique solution of the operator equation

$$\exp \mathbf{K} - \frac{\mathbf{F} \exp \mathbf{K} \mathbf{E} - \sinh \mathbf{K}}{\mathbf{I}} = 0$$

$\mathbf{F}_j$ being unitaries) hence

$$e^{\mathbf{K}} - e^{\mathbf{K}} = \int_0^t |e^{t \mathbf{K}_1} (e^{t \mathbf{K}_2 - t \mathbf{K}_1})| dt.$$。

The above theorem shows that

$$|\text{U}(\mathbf{F}, \mathbf{E}) - \text{U}(\mathbf{F}, \mathbf{E})| = |\mathbf{K} - \mathbf{K}|.$$.

In case of 2-frame with $\mathbf{E} = \mathbf{E}$, let $\mathbf{B}$ be the angle matrix between $\mathbf{M}_1$, $\mathbf{M}_2$; it is the same as the of $\mathbf{M}_1$, $\mathbf{M}_2$. Hence

$$|\text{U}(\mathbf{F}, \mathbf{E}) - \text{U}(\mathbf{F}, \mathbf{E})| = |(\text{U}(\mathbf{F}, \mathbf{E}) - \mathbf{E})|$$

The last inequality follows from Theorem 5.3.

Finally we remark that all the results in this work are still valid if we have orthogonal $r$-frames on a Hilbert space.

REFERENCES


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